Essays on Utility Maximization and Optimal Stopping Problems in the Presence of Default Risk

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Abstract

In this thesis, we study stochastic control problems faced by agents in financial market when making decisions. The thesis consists of two independent parts.

In the first part, we consider a rational risk-averse investor with utility function defined on the positive real line who aims at maximizing his expected utility from terminal wealth by trading in a financial market. We combine knowledge from convex duality theory and dynamic programming principle to derive sufficient conditions on market models which ensure that the optimizers (optimal trading strategy, optimal wealth process and dual optimizer) live in suitable normed spaces. On one hand, the normed spaces provide a natural topology to investigate the stability analysis of the optimizers w.r.t. "small" misspecification in the utility functions and initial capitals, and on the other hand, they serve in *verification theorems* when identifying optimal trading strategies. In the setting of a continuous market model, we obtain stability results for the optimal wealth process and the dual optimizer in the *Emery* topology and the uniform topology on semimartingales while for the optimal trading strategy, we obtain stability results in the L^2-topology.

For sufficiently differentiable utility functions and continuous market models, we obtain a description of the optimal trading strategy in terms of the solution of a system of forward-backward stochastic differential equations (FBSDEs). The system of FBSDEs is fully coupled and the coefficients are non-Lipschitz. We provide normed spaces of the solutions to the system of FBSDEs based on conditions on the market price of risk. We also obtain results describing the behavior of the optimal trading strategy as the coefficient of relative risk aversion approaches uniformly a constant or goes to infinity.

The second part of the thesis deals with the optimal stopping problem for an agent with a reward process exposed to a default event modeled by a random time τ. Our main concern is to give a description of the solutions before and after the default event and thereby better understand the behavior of the agent in the presence of default. We show how the stopping problem can be decomposed into two individual stopping problems: one with information flow for which the default event is not visible, and another one with information flow which captures the default event. The solutions to the individual stopping problems correspond to the solution of the original optimal stopping problem respectively before and after the default event. We apply the decomposition approach to construct explicit hedging strategies for American contingent claims in a financial market consisting of an asset with continuous paths, and another asset with a jump at τ. We build on the decomposition approach for the optimal stopping problem, and the link between the theories of optimal stopping and reflected backward stochastic differential equations (RBSDEs) to derive a corresponding decomposition approach to solve RBSDEs with a jump at τ. We obtain existence of solutions to RBSDEs with Lipschitz drivers and drivers of quadratic growth.

Zusammenfassung

Gegenstand der vorliegenden Dissertation sind stochastische Kontrollprobleme denen sich Agenten im Zusammenhang mit Entscheidungen auf Finanzmärkten gegenübersehen, nämlich das erwartete Nutzenmaximierungsproblem und das optimale Stoppproblem.

Im ersten Teil wird ein rationaler risikoaverser Investor mit nichtnegativer Nutzenfunktion betrachtet, der beabsichtigt, den Nutzen aus seinem Endvormögen durch Handeln auf einem Finanzmarkt zu maximieren. Erkenntnisse der konvexen Dualitätstheorie und der dynamischen Programmierung werden kombiniert, um hinreichende Bedingungen über Marktmodelle abzuleiten, die garantieren, dass die verwendeten Optimierer (optimale Handelsstrategie, optimales Vermögen und duale Optimierer) in geeigneten normierten Vektorräumen liegen. Einerseits bieten diese normierten Vektorräume eine natürliche Umgebung für die Stabilitätanalyse der verwendeten Optimierer für kleine Fehlspezifikationen der Nutzenfunktion und des Anfangskapitals. Andererseits sind sie nützlich in Verifikationsaussagen zur Identifikation der optimalen Handelsstrategie. Im Bereich zeitstetiger Marktmodelle erhalten wir die Stabilität des optimalen Vermögens und des dualen Optimierers in der Emery Topologie und der Topologie gleichmäßiger Konvergenz für Semimartingale, währens sich für die optimale Handelsstrategie Stabilitätsaussagen in der L^2-Topologie ergeben.
Für hinreichend differenzierbare Nutzenfunktionen und zeitstetige Martkmodelle erhalten wir eine Beschreibung der optimalen Handelsstrategie durch die Lösung eines Systems von stochastischen Vorwärts-Rückwärts-Differentialgleichungen (FBSDEs). Das System ist vollständig gekoppelt und die Koeffizienten nicht Lipschitz stetig. Wir bestimmen normierten Vektorräume für die Lösungen des Systems von FBSDEs aus Bedingungen an den Marktpreis des Risikos. Daneben erhalten wir Ergebnisse, in denen das Verhalten der optimalen Handelsstrategie beschrieben wird, wenn der Koeffizient der relativen Risikoaversion gleichmäßig gegen eine Konstante oder gegen unendlich strebt.

Der zweite Teil der Arbeit beschäftigt sich mit dem Problem des optimalen Stoppens für einen Agenten, dessen Ertragsprozess von einem Ausfallsereignis abhängt, das durch eine zufällige Zeit τ beschrieben wird. Unser Hauptinteresse gilt der Beschreibung der Lösungen vor und nach dem Ausfallsereignis, und damit dem besseren Verständnis des Verhaltens des Agenten bei Vorlage eines Ausfallsereignisses. Wir zeigen, wie das Problem des optimalen Stoppens sich in zwei individuelle Unterprobleme zerlegen lässt: eines, für das der zugrunde liegende Informationsfluss das Ausfallereignis nicht kennt, und eines, für das es den Informationsfluss einschließt. Die Lösungen der individuellen Stoppprobleme entsprechen hierbei der Lösung des ursprünglichen optimalen Stoppproblems vor beziehungsweise nach dem Ausfallreignis. Anschließend verwenden wir diesen Zerlegungsansatz zur Konstruktion expliziter Handelsstrategien für amerikanische Derivate in einem Finanzmarkt, der aus einer Anlagemöglichkeit mit stetiger Entwicklung sowie einer Zweiten mit einem Sprung zum Zeitpunkt τ besteht. Aufbauend auf dem Zerlegungssatz für das Problem des optimalen Stoppens und der Verbindung zwischen den Theorien des optimalen Stoppens und reflektieren stochastischen Rückwärts-Differentialgleichungen (RBSDEs) leiten wir einen entsprechenden Zerlegungsansatz her, um RBSDEs mit Sprung bei τ zu lösen. Wir erhalten Lösungen von RBSDEs mit Lipschitz-stetigen Erzeugern und solchen mit quadratischem Wachstum.

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To the Medjoudem Family

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1. Introduction

In this thesis, we study stochastic control problems faced by agents in financial markets when making decisions, e.g. the utility maximization problem and the optimal stopping problem.

The presence of uncertainty in financial markets prompts agents to make decisions on the basis of some optimality criteria which reflects their attitude towards risk and their risk-preferences. For a rational risk-averse agent with risk preference modeled by a utility function, and interested only in investing in the traded assets, his actions are encoded by the trading strategy yielding the maximal expected utility from terminal wealth. Of utmost concern to the agent is a description of the optimal trading strategy that is amenable to numerical approximation, so he can implement his course of actions. Another concern is the stability analysis of optimal trading strategy w.r.t. "small" misspecification in his utility function. Often, agents do not know their utility function exactly. Therefore, for the utility maximization problem to be useful for agents to plan their actions, it is important that misspecification in the utility function does not affect "too much" the resulting optimal trading strategy.

In the first part of this thesis, we combine knowledge from convex duality theory and about the dynamic programming principle to derive sufficient conditions on market models which ensure that the optimizers (optimal trading strategy, optimal wealth process and dual optimizer) live in suitable normed spaces. On one hand, the normed spaces provide a natural topology to investigate the stability analysis of the optimizers , and on the other hand, they serve in *verification theorems* when identifying the optimal trading strategy. The sufficient conditions on the market models amount to reverse Hölder's inequalities for the density of an equivalent martingale measure for the market model. They hold for exponential Lévy models while for continuous market models they reduce to a BMO-condition on the market price of risk. In the setting of a continuous market model, we obtain stability results for the optimal wealth process and dual optimizer in the *Emery* topology and uniform topology on semimartingales while for the optimal trading strategy, we obtain stability results in the L^2-topology. For sufficiently smooth utility function and continuous asset price process, we obtain a description of the optimal trading strategy in terms of the solution of a system of forward-backward stochastic differential equations (FBSDEs). We provide normed spaces of the solutions to the system of FBSDEs based on conditions on the market price of risk. We also obtain results describing the behavior of the optimal trading strategy as the coefficient of relative risk aversion approaches uniformly a constant or goes to infinity.

One particular risk firms account for is the *default risk*. Default risk is the risk that a party signed up to a contract does not meet its obligations. Default events are a common feature in financial markets, and they affect prices of assets and credit ratings of firms having an exposure to the default events. For firms facing stochastic control problems in the presence of default risk, an important concern is to have a description of the solutions before and after the default event triggering the risk. The knowledge of solutions before and after the default event gives a better understanding of the impact of the risk, and allows to build efficient hedging strategies against such a risk. In particular, it provides a precise description of the course of actions of firms before and after the default event.

The second part of the thesis deals with the optimal stopping problem with a reward process exposed to a default event modeled by a random time τ. The optimal stopping problem consists in choosing a stopping time ν such that the expected value of the reward process evaluated at ν is maximal among all possible choices of stopping times. We show how the optimal stopping problem in the presence of default risk can be decomposed into two individual stopping problems: one in a filtration for which the default event is not visible, and another in a filtration which captures the default event. The solutions to the individual stopping problems correspond to the solution of the original optimal stopping problem respectively before and after the default event. Our decomposition approach is based on techniques from filtration enlargements and the *dynamic programming principle*. We apply the decomposition approach to construct explicit hedging strategies for American contingent claims in a financial market consisting of an asset with continuous paths, and another asset with a jump at τ. We build on the decomposition approach for the optimal stopping problem, and the link between the theories of optimal stopping and reflected backward stochastic differential equations (RBSDEs) to derive a corresponding decomposition approach to solve RBSDEs with a jump at τ. We obtain existence of solutions to RBSDEs with Lipschitz drivers and drivers of quadratic growth.

In the sequel, we give a formulation of the aforementioned problems, a brief overview of the related literature and a more precise description of our main results.

1.1 Problem formulations and existing literature

1.1.1 Utility maximization problem

We consider a risk averse investor endowed with an initial capital $x > 0$ and trading in a financial market with asset price processes modeled by a semimartingale S. The risk preference of the investor is given by a utility function U, i.e. a strictly increasing, strictly concave and continuously differentiable function. The investor is assumed to be *small* in the sense that his actions do not influence the dynamics of the asset prices. The goal of the investor is to look among all admissible trading strategies available to him, for a trading strategy which maximizes his expected utility from terminal wealth. Hence, the primal stochastic control problem with value function u given by

$$u(x) \quad := \quad \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[U(X_T^\pi)\right], \tag{1.1}$$

where X_T^π represents the terminal wealth obtained from trading in a self-financing way according to π, T the trading time horizon and $\mathcal{A}(x)$ the set of self-financing trading strategies admissible for the initial capital x. A trading strategy attaining the sup in (1.1) is referred to as an optimal trading strategy and the corresponding wealth, the optimal wealth process. The stochastic control problem (1.1) was first addressed by [Mer69, Mer71], and has been extensively studied in the literature, see [KLSX91, Sch04] and references therein.

The main approach to address the existence of an optimal trading strategy for a fairly general utility function U and market model S is by method of *convex duality*, see [KLSX91, KS99, KS03]. A key requirement of this approach is the absence of arbitrage which constitutes in essence a fairness condition for the market model in the sense that one cannot make profits from trading on the assets without taking any risk. By a Fundamental Theorem of Asset Pricing (FTAP), the absence of arbitrage is equivalent to the existence of so-called *equivalent local martingale measures*, see [DS94]. The latter are probability measures that equivalent to the underlying real world measure, and under which asset prices are local martingales. In the convex duality approach, one looks at a *dual problem* associated to the problem (1.1) which amounts to a minimization problem with objective function given by a functional of the convex conjugate of U. The dual domain consists of the set of *supermartingale deflators* for S which is an enlargement of the set of densities of equivalent local martingale measures for S. Under the so-called reasonable asymptotic elasticity assumption on U which ensures that U has power growth for large values, the dual problem admits a solution called the *dual optimizer* $\widehat{Y}$. The existence of a solution to (1.1) is then retrieved via standard arguments from convex analysis. One has the following *duality pairing* between the optimal wealth process $\widehat{X}$ and dual optimizer $\widehat{Y}$: $\widehat{Y}_T = U'(\widehat{X}_T)$ and $\widehat{X}\widehat{Y}$ is a true martingale. Convex duality was first adopted in [KLS87, Pli86] in a setting of complete market models, i.e. under the existence of a unique equivalent local martingale measure. [KLS87] then extended the approach to Itô process models. General semimartingale models have been completely treated in [KS99, KS03].

Though the existence of optimal trading strategies is guaranteed under simple assumptions on the market model and utility function, an explicit description of optimal strategies in the setup of incomplete market models which is amenable to numerical computations is available only for the classical utility functions: *logarithmic utility* $U(z) = \log z$, *power utilities* $U(z) = \frac{z^p}{p}$, where $p \in (-\infty, 0) \cup (0, 1)$ and *exponential utilities* $U(z) = -\exp(-\alpha z)$ with $\alpha > 0$. In the first part of this thesis, we focus on the description of optimal trading strategies for utility functions defined on the positive real line and continuous market models, and their stability analysis w.r.t. to misspecification in the utility function and initial capital.

For the classical utility functions, optimal trading strategies are given by means of solutions to backward stochastic differential equations (BSDEs). Before explaining the approaches leading to the description of optimal trading strategies in the general case, we give a short overview of the theory of backward stochastic differential equations.

Backward stochastic differential equations

On a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ generated by a Brownian motion B, a BSDE is an equation of the form

$$Y_t \;=\; \xi + \int_t^T F(s, Y_s, Z_s) ds - \int_t^T Z_s dB_s, \; t \in [0, T], \tag{1.2}$$

where $F : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is a predictable mapping called the *driver*, $\xi : \Omega \to \mathbb{R}$ an $\mathcal{F}_T$-measurable random variable called the *terminal value* and T the terminal horizon. Solving a BSDE consists in finding a pair of adapted processes (Y, Z) which satisfies the equation (1.2). A BSDE therefore describes the dynamics of a *controlled semimartingale* Y with control variable Z and predetermined terminal value ξ. The control variable Z ensures that Y meets the adaptedness requirement and the value ξ at terminal time.

Backward stochastic differential equations (BSDEs) were introduced by [Bis73] to describe the dynamics of *adjoint equations*[1] appearing in the *stochastic maximum principle* for optimal control problems with controlled state variables driven by an Itô dynamics. There have been a great interest in the investigation of BSDEs. In a Brownian setting, [PP90] proved existence and uniqueness of a square integrable solution if F is globally Lipschitz continuous in (Y, Z) and the standard parameters (F, ξ) are square integrable. The article [EKPQ97b] provides a survey of applications of BSDEs with Lipschitz drivers in mathematical finance including for example pricing, hedging and superreplication of contingent claims. For drivers having quadratic growth in the control variable Z, and bounded terminal value, the existence of a solution (Y, Z) with uniformly bounded Y has been obtained by [LSM98, Kob00]. Similar results have also been obtained in the setup of a continuous filtration, i.e. a filtration w.r.t. which all local martingales have continuous paths, see [Tev08, Mor09a]. The latter results pave the way for the application of BSDEs to the utility maximization problem for classical utility functions (logarithmic, power and exponential utilities) and asset price processes having continuous paths, see [EKR00, HIM05, Sek06, Mor09a]. Existence results for BSDEs with drivers of quadratic growth in Z, and terminal value having exponential moments of a certain order have also been investigated: in the Brownian setting by [BH08, DHR11] and in the case of a continuous filtration by [MW12, BEK13]. For stability of solutions w.r.t. standard parameters, see [Fre13, MW12]. Stability properties of BSDEs have become a key tool in the analysis of robustness properties of solutions to stochastic optimal control problems w.r.t. input parameters.

[1] Adjoint equations are processes acting as dual variables to the controlled state variables.

This has been shown for example in the cases of utility maximization problems for power and exponential utilities [MW13, Fre13]. We will rely on stability theorems for BSDEs in Chapter 3 to describe the asymptotic behavior of solutions to the utility maximization problem w.r.t. relative risk aversion. Numerical schemes for BSDEs have been also studied in the literature, see [BT04, Zha04, BZ08, IDRZ10, CR16].

Though BSDEs have been well studied in a self-contained way, for applications in mathematical finance they often arise in systems of coupled forward and backward stochastic differential equations (FBSDEs). On a Brownian basis, FBSDEs have the following form

$$X_t = x + \int_0^t b(s, X_s, Y_s, Z_s)ds + \int_0^t c(s, X_s, Y_s, Z_s)dB_s, \tag{1.3}$$

$$Y_t = h(X_T) + \int_t^T F(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dB_s, \; t \in [0, T], \tag{1.4}$$

where b, c, h and F are measurable functions which constitute their input data. Here a solution consists of a triple of adapted processes (X, Y, Z) satisfying (1.3) and (1.4). The existence of solutions to fully coupled systems of FBSDEs has been mainly investigated for globally Lipschitz continuous data b, c, F and h. We refer to the monograph [MY99] for a survey of approaches of existence of solutions. For recent results on FBSDEs, we refer to [MWZZ15, FI13, LL17]. In Chapter 3, we will obtain formulas of optimal trading strategies to the utility maximization problem by means of solutions to a fully coupled system of non-standard FBSDEs. The system is non-standard in the sense that the driver is not Lipschitz, and properties of the solutions cannot be obtained from the existing literature on FBSDEs. We will provide an analysis of the spaces in which the solutions live on the basis of an appropriate set of *a priori* estimates derived using the dynamic programming principle.

Explicit solutions to the utility maximization problem

For complete market models, the solution to the dual problem to (1.1) has an explicit formula in terms of the density of the unique equivalent local martingale measure. The duality pairing between the dual optimizer and the optimal wealth process leads to an explicit formula of the optimal wealth from which one can then derive the optimal trading strategy, see [KLSX91]. In the setup of incomplete markets, the solution to the dual problem appears unknown. As a result, optimal trading strategies are determined using direct stochastic control approaches of the dynamic programming principle [EK81] or the stochastic maximum principle [Pen93].

The approach of the dynamic programming principle (DPP) is built on the characterization of optimality through value processes. To a trading strategy π, one associates a value process $u(\cdot, X^\pi)$ defined for each time t as the maximal conditional expected utility that can be achieved on the remaining time interval $[t, T]$, assuming that the trading strategy π was followed up to time t. More precisely,

$$u(t, X_t^\pi) = \operatorname*{ess\,sup}_{\theta \in \mathcal{A}_t(x,\pi)} \mathbb{E}\left[U(X_T^\theta)|\mathcal{F}_t\right], \tag{1.5}$$

where $\mathcal{A}_t(x, \pi) = \{\theta \in \mathcal{A}(x) \mid \theta_s = \pi_s, s \in [0, t]\}$. By the DPP, the value process associated to every trading strategy is a supermartingale, and a trading strategy is optimal if and only if the associated value process is a martingale. Hence identifying optimal trading strategies amounts to the knowledge of the canonical decompositions of value processes in order to determine the ones which are true martingales. In the particular case of power utility functions, i.e. $U(z) = \frac{z^p}{p}$, $p \in (-\infty, 0) \cup (0, 1)$, the homogeneity of power functions attributes to the value processes a multiplicative decomposition into the utility of the current wealth and a process $L(p)$ independent of the initial capital. One refers to $L(p)$ as the *opportunity process* as it describes the maximal conditional expected utility that can be obtained at each time from unit capital, see [Nut10]. The knowledge of the value process reduces to that of $L(p)$. The application of the DPP leads to the description of the dynamical behavior of $L(p)$ by a backward stochastic differential equation, and to a formula of optimal trading strategies in terms of the canonical decomposition of $L(p)$, see [MT03b, Nut12a, MT03a]. For logarithmic and exponential utilities, value processes admit also a decomposition into the utility of the current wealth and a corresponding *opportunity process* which describes optimal trading strategies, see [EKR00, MT03b, MT03a]. The approach of DPP was first used in [EKR00, Sek06, HIM05] in the setting of a Brownian filtration. It was then extended to continuous filtrations by [Mor09a, MS05]. For the cases of market models with jumps and general filtrations, we refer to [Bec06, Mor09b, Nut12a]. For utility functions different from classical utilities, it is not always possible to obtain a description of the dynamical behavior of the value processes without further assumptions on these and the market model. We refer to [MT10] for an attempt to describe value processes using the theory of backward stochastic partial differential equations. For asset price processes with Markovian dynamics, value processes can be explicitly determined using the solution to the Hamilton-Jacobi-Bellman equation satisfied by the value function, see [KS98].

An alternative approach to identify optimal strategies for stochastic control problems is the stochastic maximum principle developed by [Pen93, Pen90]. It only requires an Itô dynamics for controlled state processes, and it leads to a *first order optimality condition* in terms of the corresponding *adjoint equations*. For U defined on the positive real line and sufficiently smooth, and continuous S, [HHI$^+$14] used the approach of the stochastic maximum principle for the utility maximization problem (1.1) in the setup of a Brownian basis. The key insight obtained in [HHI$^+$14] leading to explicit descriptions for optimal trading strategies is that the dual optimizer $\widehat{Y}$ solves the adjoint equation, and it admits a factorization into the marginal utility of the optimal wealth $\widehat{X}$ and a process L depending only on $\widehat{X}$. More precisely,

$$\widehat{Y} = U'(\widehat{X})L. \tag{1.6}$$

Using similar arguments leading to the *first order optimality condition* in [Pen93], [HHI$^+$14] derived a system of forward-backward stochastic differential equations (FBSDEs) describing the joint dynamics of $(\widehat{X}, L)$. The forward equation of the system of FBSDEs describes the optimal wealth process $\widehat{X}$, while the backward component describes the dynamics of L. Optimal trading strategies are given in terms of L and its canonical decomposition. In the particular case of logarithmic utility, L is given by the constant 1 and the optimal trading strategy depends only on the market price of risk of the asset. For U of power type, L coincides with the opportunity process, see [Nut10, MT03b]. For this reason, we will call L the *generalized opportunity process*.

An important study not addressed in [HHI$^+$14] is that of spaces where the solutions to the system of fully coupled FBSDEs describing the joint dynamics of $(\widehat{X}, L)$ live in. The knowledge of the normed spaces for the solutions turned out to be a crucial property for proving *verification theorems*. In principle, L is not known explicitly except for logarithmic utility. Hence, to determine an optimal trading strategy, one needs to solve the system of FBSDE describing the joint dynamics of $(\widehat{X}, L)$ to find a candidate for an optimal trading strategy, and then verify if the candidate is indeed an optimal strategy. The verification step often reduces to showing that the stochastic exponential of a local martingale depending on the martingale part of the solution to the FBSDEs is a true martingale. As a result, the step requires additional knowledge about the normed spaces associated to the solutions. For the class of power utilities, the most common requirement guaranteeing that a solution of the FBSDEs describing the joint dynamics of $(\widehat{X}, L)$ will lead to an optimal trading strategy is that the backward component of the solution be uniformly bounded, see [HIM05, Mor09a]. For the latter class, L has been well studied in the setup of a general semimartingale market model S, and necessary and sufficient conditions on the model primitives ensuring that L is uniformly bounded are well understood, see [MT03b, Nut10, FMW12]. In Chapter 2, we will provide for a general utility function U and semimartingale model S *a priori* estimates of L which will enable us to derive necessary and sufficient conditions for L to belong to suitable normed spaces, e.g. that L is uniformly bounded. We will rely on the *a priori* estimates of L in Chapter 3 to study the normed spaces of the solutions to the system of FBSDEs describing the joint dynamics of $(\widehat{X}, L)$ under various conditions on the market price of risk. Besides verification theorems, the normed spaces associated to the solutions will also set the stage to apply BSDEs stability results for the study of the asymptotic behavior of the optimal trading strategy w.r.t. risk aversion in various topologies.

Stability analysis of the utility maximization problem

In practice, no agent in the market knows his utility function exactly. Moreover, parameters of models for asset prices are often obtained by calibration of historical data. They are thus filled with imperfections. The misspecification in the inputs of the utility maximization problem raises the question whether the solutions are robust w.r.t. the input data. The stability analysis of solutions w.r.t. risk preference and initial capital was initiated by [JN04] in the setup of a complete Itô's market model with uniform bounded market price of risk. The authors show that for a sequence of utility functions converging pointwise and satisfying a uniform growth condition , the corresponding sequence of optimal wealths converges almost surely and in L^1 at every date. They also obtain an L^1-convergence result for the resulting sequence of optimal trading strategies. The strong convergence result is mainly due to the assumed completeness of the market and boundedness of the market price of risk. These lead to explicit formulas and nice integrability properties for the optimal trading strategy. In [Lar09], an incomplete market model with continuous asset price is considered. The author establishes the stability of the optimal terminal wealth in the topology of convergence in probability. For a fixed utility function, [Lv07] examines the stability of the optimal terminal wealth in the topology of convergence in probability. The case of optimal wealth at a stopping time τ with values in $[0, T]$ is studied by [BK10]. A unified treatment of the stability properties of the optimal terminal wealth in the context of general semimartingale models for the asset w.r.t. misspecification in risk preference, underlying probability measure, initial capital and even w.r.t. positions in an illiquid asset have been obtained in [KŽ11]. In stability is also w.r.t. convergence in probability.

The above references rely on general convex duality techniques. While they allow to deal with general utility functions, the focus is on optimal terminal wealth. Convergence in probability is too weak to provide insight on the stability of optimal trading strategies without further assumptions. From a practical point of view, the utility maximizer is more concerned with his optimal trading strategy as it encodes the actions he undertakes to achieve his goal, and the optimal wealth process which keeps track of the success of his actions at each date.

In the present thesis, we are interested in the stability analysis of the optimizers (optimal trading strategy, optimal wealth process and dual optimizer) w.r.t. misspecification in the utility function and initial capital. The optimizers being stochastic processes, a primary step to the analysis is to identify a suitable topology in the spaces in which the optimizers live. In the setup of a general semimartingale, we will derive in Chapter 2 sufficient conditions on the model which guarantee that the optimizers are embedded into normed spaces providing an appropriate topology for our analysis. In the setting of a continuous asset price model, we will derive stability results for the optimal wealth process and dual optimizer in the Emery topology and the uniform topology on semimartingales. For the optimal trading strategy, we will obtain stability w.r.t. the L^2-topology.

1.1.2 Stochastic control problems in a progressively enlarged filtration

An important topic in probability theory studied in the 70's and 80's, and also of pure mathematical interest is that of filtration enlargements, see [JY78, Jeu79, Jeu80, Jac85] and references therein. Given a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ and a random variable τ, two types of enlargements of $\mathbb{F}$ by τ were considered: the *initial enlargement* of $\mathbb{F}$ by τ, and in case τ is positive, the progressive enlargement of $\mathbb{F}$ by τ. In the latter case, τ usually describes a random time. The initial enlargement $\mathbb{G}^\tau = (\mathcal{G}_t^\tau)_{t\geq 0}$ of $\mathbb{F}$ by τ is the smallest right continuous filtration containing $\mathbb{F}$ and such that τ is $\mathcal{G}_0^\tau$-measurable, while the progressive enlargement $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ of $\mathbb{F}$ by τ is the smallest right continuous filtration containing $\mathbb{F}$ and making τ a $\mathbb{G}$-stopping time. The σ-algebras underlying the filtrations $\mathbb{G}^\tau$ and $\mathbb{G}$ have the following form

$$\mathcal{G}_t^\tau = \cap_{s>t}\left(\mathcal{F}_s \vee \sigma(\tau)\right) \text{ and } \mathcal{G}_t = \cap_{s>t}\left(\mathcal{F}_s \vee \sigma\left(\tau\wedge u,\ u\leq s\right)\right).$$

Works on filtration enlargements initially focused on canonical decompositions of $\mathbb{F}$-semimartingales in the enlarged filtrations for which the semimartingale property is preserved. A by now well known sufficient condition ensuring the preservation of the semimartingale property is the absolute continuity of the regular conditional distribution of τ given $\mathbb{F}$ w.r.t. the law of τ also known as *Jacod's hypothesis*. Jacod's hypothesis also allows to derive explicitly the corresponding canonical decomposition of $\mathbb{F}$-semimartingales in the enlarged filtrations. It was first introduced for the study of initial enlargements by [Jac85], and then adopted for progressive enlargements by [JLC09b]. In recent years, there has been an increase of interest in the description of martingales and measurability properties w.r.t. the enlarged filtrations. This interest is motivated by applications in financial modeling that we discuss below. A dominant hypothesis in this line of research is the *density hypothesis*. It strengthens *Jacod's hypothesis* by assuming equivalence of the regular conditional distribution of τ given $\mathbb{F}$ w.r.t. the law of τ, see [FI93, GP98, Ame00]. Under the density hypothesis, the preservation of the *strong predictable representation property* in passing from $\mathbb{F}$ to the enlarged filtrations holds, see [Ame00, CJZ13]. Moreover, local martingales in the enlarged filtrations can be described uniquely in terms of their counterparts in the filtration $\mathbb{F}$ as shown recently by [Fon15, EKJJ10, CJZ13]. For the initial enlargement $\mathbb{G}^\tau$, [Fon15] shows that optional processes can be identified with a family of indexed $\mathbb{F}$-optional processes. Regarding optional processes in the progressive enlargement $\mathbb{G}$, [Son14] obtained a characterization known as the *optional splitting formula*: for every $\mathbb{G}$-optional process ψ, there exist an $\mathcal{O}(\mathbb{F})$-measurable process ψ^{b} and an $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable process $\psi^{\mathrm{d}}(\cdot)$ such that for every $t\geq 0$

$$\psi_t = \psi_t^{\mathrm{b}} 1_{\{t<\tau\}} + \psi^d(\tau) 1_{\{t\geq\tau\}}. \tag{1.7}$$

The processes ψ^{b} and $\psi^d(\tau)$ in the above decomposition are referred to as the pre-default and post-default values of ψ. A characterization of stopping times has also been obtained in [EI18, BZ14].

New interest in filtration enlargements arose from applications in mathematical finance as a toolbox for modeling phenomena in financial markets such as insider trading [GP97, GP98, PK96] and default risk [Lan98, Kus99, EJY00, BJR04, BR02]. In both contexts, $\mathbb{F}$ models the information flow that is publicly available to all agents and usually consists of spotted prices of assets, interests rates, vanilla options, etc. It is often referred to as the *reference filtration*. The filtration $\mathbb{G}^\tau$ serves as toy model of the information flow of an *informed agent* with additional information given by τ, e.g. the running maximum at maturity of the price of a certain asset. The filtration $\mathbb{G}$ is a standard model for *default risk modeling*. In this context, τ is the time a surprise default event takes place, e.g. the downgrading of the creditworthiness of a firm, and $\mathbb{G}$ models the global market information flow including the progressive knowledge of the occurrence of the event. This is for example the relevant information flow of an agent with an exposure to the default event. For filtrations other than initial and progressive types, we refer to [Ank05, Kch11].

In this thesis, we are mainly concerned with the progressive enlargement $\mathbb{G}$ of a reference filtration $\mathbb{F}$ by a random time τ. In the context of default risk modeling, when dealing with control problems it is important to identify the pre-default and post-default values of solutions in order to understand the impact of the default event and to design efficient risk management strategies. This is particularly relevant in the situation of successive default events where one is interested in the impact of first default event on the following ones. The identification of pre-default and post-default values of solutions to control problems gives the possibility to obtain a suitable description of the actions of the agents facing the control problems before and after the default event, and may also lead to efficient numerical implementation of the solutions. This fact has been illustrated recently for several control problems. These include for examples the utility maximization problem with defaultable securities [JP11, Pha10, JKP13, IJL16, CFL14] and mean-variance hedging of defaultable claims [CGN15].

A direct application of standard approaches to address control problems in the filtration $\mathbb{G}$ such as the dynamic programming principle does not lead to the description of pre-default and post-default values of solutions. The description relies on a so-called *decomposition approach* introduced by [JP11, Pha10] in the context of the utility maximization problem. The approach consists in exploiting the splitting formulas of wealth processes and formulas for computing conditional expectations to reduce the initial problem into two control sub-problems of similar structure in the filtration $\mathbb{F}$: an after-default problem parametrized by the occurrence of default and a global pre-default problem depending on the after-default problem. The solutions to the after-default and pre-default problems correspond respectively to the post-default and pre-default values of the solution to the initial utility maximization problem. The approach extends naturally to multiple default events [JKP13] and has been successfully applied to address other stochastic control problems such as mean-variance hedging [CGN15] and controller-stopper problems [BZ14]. A similar *decomposition approach* to construct bounded solutions to BSDEs has been studied in [ABSEL10, KL12] in the setup where $\mathbb{F}$ is the natural filtration of a Brownian motion. In the latter case, the decomposition approach reduces the solvability of the original BSDEs to that of a recursive systems of Brownian BSDEs.

In the second part of this thesis, we will adopt the decomposition approach to give an explicit description of the pre-default and post-default values of solutions to the optimal stopping problem and reflected backward stochastic differential equations (RBSDEs). In the sequel, we give the formulation of the optimal stopping problem and an overview of the existing literature on RBSDEs.

Optimal stopping problem

We consider an agent endowed with a reward modeled by an $\mathbb{F}$-adapted càdlàg process X. The optimal stopping problem for the agent consists in finding an $\mathbb{F}$-stopping time ν valued in $[0, T]$ for which the expected valued of X evaluated at ν is maximal among all possible choices of $\mathbb{F}$-stopping times valued in $[0, T]$, i.e.,

$$\sup_{\gamma \in \mathcal{T}_{0,T}(\mathbb{F})} \mathbb{E}\left[X_\gamma\right] = \mathbb{E}\left[X_\nu\right], \tag{1.8}$$

where for an $\mathbb{F}$-stopping time ς, $\mathcal{T}_{\varsigma,T}(\mathbb{F})$ denotes the set of $\mathbb{F}$-stopping times valued in $[\varsigma, T]$. The optimal stopping problem dates back to 40's and has been extensively studied due to its applications in areas such as statistics, partial differential equations, stochastic analysis, finance, economics, etc. We refer to [PS06] for various constructions of optimal stopping times in a Markovian setup, and to [Mai78, EK81] for the general case.

The characterization of optimal stopping times for the problem (1.8) is based on the *dynamic programming principle* and the *Snell envelope* of X. The Snell envelope describes the maximal conditional expected value that can be achieved from a given time considered as initial time. It acts as the dynamic value process associated to (1.8), and has the characteristic feature that it is the smallest supermartingale that dominates the process X. It is defined by

$$V_\varsigma = \operatorname*{ess\,sup}_{\gamma \in \mathcal{T}_{\varsigma,T}(\mathbb{F})} \mathbb{E}\left[X_\gamma | \mathcal{F}_\varsigma\right], \ \varsigma \in \mathcal{T}_{0,T}(\mathbb{F}). \tag{1.9}$$

A stopping time is known to be optimal if and only if the Snell envelope stopped there is a martingale, and the Snell envelope coincides with the reward at the stopping time, see [EK81]. Solving the optimal stopping problem (1.8) reduces to the knowledge of the Snell envelope.

The optimal stopping problem arises in finance when pricing American contingent claims. An American contingent claim (ACC) is a financial contract with payoff modeled by a non-negative adapted process, and with the specific feature that the contract can be exercised at any time up to the trading horizon. In the setup of complete market models for asset price processes, it has been established that the price of an American contingent claim is given by the value of the optimal stopping problem of the associated reward w.r.t. the unique equivalent local martingale measure for the asset price process. A hedge for the ACC can be constructed using the martingale part of the Snell envelope and the martingale representation theorem w.r.t. the underlying asset price under the unique equivalent local martingale measure, see [Ben84, Kar88, KK98, KS98]. Another application of the optimal stopping problem comes from its connection to RBSDEs (to be defined below) via the Snell envelope. Solutions to such equations can be represented as Snell envelopes of suitably chosen processes, and thus interpreted as the values of an optimal stopping problem.

The optimal stopping problem in the filtration $\mathbb{G}$ has been considered if the reward is an $\mathbb{F}$-adapted process stopped at τ in [Szi05, BCJR09], and with a focus only on the pre-default values of optimal stopping times. [BZ14] consider the more general controller-stopper problem in the multiple default setting for which pre-default and post-default values of an optimal stopping time are constructed using solutions to reflected BSDEs. However, the necessity for optimality is not addressed and the reward is assumed to have continuous paths between two successive default events. The complete description of pre-default and post-default values of optimal stopping times require the precise knowledge of the pre-default and post-default values of Snell envelopes. In Chapter 4, we will provide the description of the pre-default and post-default values of Snell envelopes using a characterization of $\mathbb{G}$-supermartingales in terms of $\mathbb{F}$-supermartingales and $\mathbb{G}^\tau$-supermartingales, and control arguments. The description of Snell envelopes will enable us to derive the full characterization of pre-default and post-default values of optimal stopping times for the optimal stopping problem for a fairly general reward.

Reflected backward stochastic differential equations

Reflected backward stochastic differential equations (RBSDEs) are backward stochastic differential equations for which the solution is required to stay above a certain process, called *obstacle*. In the setup of a Brownian filtration $\mathbb{F} = (\mathcal{F}_t)_{t \leq T}$ and a continuous obstacle process , RBSDEs have the form

$$\begin{cases} Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s, \\ Y_t \geq S_t, \ t \in [0,T], \\ \int_0^T (Y_s - S_s)dK_s, \end{cases} \tag{1.10}$$

where besides the terminal vlaue ξ and the driver F having the same significance as for BSDE the obstacle process S satisfying $S_T \leq \xi$ appears. A solution to the RBSDE with driver F, obstacle S and terminal value ξ is a triplet of adapted processes (Y, Z, K) with a predictable increasing process K starting at 0 and such that the equation (1.10) holds for each t. The role of the increasing process K is to make sure that the value process Y stays above the obstacle S and this in a minimal way encoded by the *Skorohod condition* $\int_0^T (Y_s - S_s)dK_s = 0$.

RBSDEs were introduced by [EKKP+97] and shown to be related to the optimal stopping problem and obstacle problems for nonlinear PDEs. The bridge between RBSDEs and optimal stopping problems is given by the Snell envelope representation of their solutions. Namely for a solution (Y, Z, K), one has

$$Y_\varsigma \quad = \quad \operatorname*{ess\,sup}_{\gamma \in \mathcal{T}_{\sigma,T}(\mathbb{F})} \mathbb{E}\left[\xi_\gamma 1_{\{\gamma=T\}} + S_\gamma 1_{\{\gamma<T\}} + \int_\varsigma^{T\wedge\varsigma} F(s, Y_s, Z_s)ds \big| \mathcal{F}_t\right], \; \varsigma \in \mathcal{T}_{0,T}(\mathbb{F}). \quad (1.11)$$

RBSDEs are a tailor made tool to derive solutions to stochastic control problems involving stopping times as control variables. These include for example the pricing and hedging of American contingent claims in incomplete markets [KLQT02, LMX05], and mixed optimal and risk sensitive mixed optimal problems, see [Ham02]. [EKKP+97] establish existence and uniqueness of square integrable solutions for square integrable data (F, S, ξ) and globally Lipschitz drivers F. In [KLQT02, Lio14], the authors establish existence of solutions for drivers having superlinear growth in Y, quadratic growth in Z, and bounded obstacle S and terminal value ξ. The results by [KLQT02] have been extended to the case of unbounded terminal value and/or unbounded obstacle, see [LX07, BY12]. Obstacles with càdlàg paths and globally Lipschitz drivers have been studied in [Ham02, LX05]. For càdlàg paths, the Skorohod condition takes the form $\int_0^T (Y_{t-} - S_{t-})dK_t = 0$. The case with optional obstacles have also been investigated, see [GIO+15]. Another stream of research deals with data in $\mathbb{L}^p$ for $p \in [1, +\infty)$ i.e. $\mathbb{E}\left[\left(\int_0^T |F(s,0,0)|ds\right)^p\right] + \mathbb{E}\left[\sup_{t\in[0,T]} |S_t|^p\right] + \mathbb{E}\left[|\xi|^p\right] < +\infty$, see [Ama09, HP12, Kli12, RS12]. Beyond continuous filtrations, existence and uniqueness of solutions have also been derived for filtrations generated by a Brownian motion and an independent Poisson process, in case of Lipschitz continuous drivers, square integrable data and càdlàg obstacles, see [HO16, Ess08, QS14]. The case of a general filtration (complete and right continuous) supporting a Brownian motion has also been considered in [Kli15, BPTZ15]. In this case RBSDEs take the form

$$\begin{cases} Y_t = \xi + \int_t^T F(s, Y_{s-}, Z_s)ds + K_T - K_t - \int_t^T Z_s dB_s - M_T + M_t, \\ Y_t \geq S_t, \;\; t \in [0,T], \\ \int_0^T (Y_{t-} - S_{t-})\, dK_t = 0. \end{cases} \quad (1.12)$$

A solution now consists of a quadruple of processes (Y, Z, M, K) with a local martingale M orthogonal to B, i.e. a martingale M for which the covariation process $[B, M] = 0$ vanishes. For data (F, S, ξ) in $\mathbb{L}^p$ for some $p > 1$, [Kli15] establishes the existence and uniqueness of solution for F monotonic and depending only on Y while [BPTZ15] treat the case in which F is globally Lipschitz in (Y, Z). A key difficulty when working with data in $\mathbb{L}^p$ for $p \in (1, 2)$ is the derivation of a priori estimates necessary for the construction of an approximating sequence for a solution, see [BDH+03, KP15, Kli13]. An additional difficulty arising in the scenario of a general filtration is the control of the process $[M, K]$ which does not vanish as M and K may jump at the same time. This has led to a new type of a priori estimates for proving existence of solutions, see [BPTZ15]. In Chapter 5, we will consider RBSDEs in the setting of the progressively enlarged filtration $\mathbb{G}$. If one assumes that the reference filtration $\mathbb{F}$ supports an $\mathbb{R}^n$-valued Brownian motion B, then the general form of a reflected BSDE in $\mathbb{G}$ has the form

$$\begin{cases} Y_t = \xi + \int_t^T F(s, Y_{s-}, Z_s, U_s)ds + K_T - K_t - \int_t^t Z_s dB_s^{\mathbb{G}} - \int_t^T U_s dN_s^{\mathbb{G}} - M_T + M_t, \\ Y_t \geq S_t, \\ \int_0^T (Y_{s-} - S_{s-})dK_s = 0, \;\; t \in [0,T], \end{cases} \quad (1.13)$$

where $B^{\mathbb{G}}$ is a Brownian motion in the filtration $\mathbb{G}$ and corresponds to the local martingale part of the canonical decomposition of B in the filtration $\mathbb{G}$, and $N^{\mathbb{G}}$ is the martingale part of the indicator process $D = \left(1_{\{t\leq\tau\}}\right)_{t\leq T}$ in the filtration $\mathbb{G}$. A solution to (1.13) is a quintuple (Y, Z, U, M, K) of $\mathbb{G}$-adapted processes with a local martingale M orthogonal to $B^{\mathbb{G}}$ and $N^{\mathbb{G}}$, and a $\mathbb{G}$-predictable increasing process K. The decomposition approach will allow us to extend the new existence result for RBSDEs in [BPTZ15] on the background of a general filtration supporting only a Brownian motion to the framework of a general filtration supporting a Brownian motion $B^{\mathbb{G}}$ and a pure jump martingale $N^{\mathbb{G}}$. We will also provide existence results for RBSDEs of quadratic growth with a single jump which have not yet been studied so far.

1.2 Summary of chapters

The results obtained in this thesis are divided into two main parts. The first part is concerned with the description and analysis of solutions to the utility maximization problem using knowledge from convex duality theory, dynamic programming principle and the theory of BSDEs. It corresponds to the first two chapters and the articles [IN17d, IN17a]. The second part addresses the optimal stopping problem and reflected BSDEs in the setup of a progressively enlarged filtration by means of the *decomposition approach*. It corresponds to the last two chapters and the articles [IN17b, IN17c]. Every chapter is self-contained and can be read independently.

Chapter 2: Utility maximization: integrability properties and stability analysis of the solution

In this chapter, we consider the utility maximization problem from terminal wealth for utility functions U defined on the positive real line, with initial capital $x > 0$, and an asset price modeled by a càdlàg semimartingale S. We combine knowledge from the convex duality approach and the dynamic programming principle to derive integrability properties for the *generalized opportunity process* L and the optimizers: optimal trading strategy, optimal wealth process $\widehat{X}$ and dual optimizer $\widehat{Y}$. We recall that $L = \frac{\widehat{Y}}{U'(\widehat{X})}$. The integrability properties of the optimizers will allow us to identify suitable topologies for the investigation of their stability analysis w.r.t. misspecification in the utility function and initial capital for continuous S. The integrability properties of L will provide a basis for the study of the spaces of solutions to the system of FBSDEs describing the joint dynamics of $(\widehat{X}, L)$.

The integrability properties of interest here are built upon the notion of *weighted norm inequalities* such as the *reverse Hölder inequality* (b_q^-) for $q \in (0,1)$ and the probabilistic *Muckenhoupt condition* (A_r) for $r > 1$. We recall that a strictly positive adapted càdlàg process Z satisfies (b_q^-) (resp. (A_r)) if there exists a constant C (resp. K) such that for every stopping time ς

$$\mathbb{E}\left[\left(\frac{Z_T^q}{Z_\varsigma^q}\right)\middle|\mathcal{F}_\varsigma\right] \geq C \left(\text{resp. } \mathbb{E}\left[\left(\frac{Z_\varsigma}{Z_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\varsigma\right] \leq K\right). \tag{1.14}$$

The weighted norm inequalities (b_q^-) and (A_r) have been well studied in the literature [Kaz94, DDM79]. A useful application is that, for the stochastic exponential of a local martingale, they guarantee that it is uniformly bounded in L^p for some $p > 1$, see [ISS79]. The basic idea to identify a topology for our investigation is to derive sufficient conditions on the market model S which ensure that optimal wealth processes and dual optimizers satisfy (b_q^-) or (A_r). The weighted norm will guarantee optimal wealth processes and dual optimizers are uniformly bounded in L^p for some $p > 1$. The latter property will then provide us with a range of topologies. The main assumption made for the utility function is the following growth condition on the relative risk aversion coefficient: there exist strictly positive constants a and b such that

$$a \leq -\frac{zU''(z)}{U'(z)} \leq b, \ \forall z > 0. \tag{1.15}$$

The literature on integrability properties for the optimizers for general semimartingale models for asset prices is quite limited and has been mainly focused on the dual optimizer. For U of power type with risk aversion parameter $p \in (-\infty, 0) \cup (0, 1)$, i.e. $U(z) = \frac{z^p}{p}, z > 0$, [Nut10] shows that if there exists a supermartingale deflator Z which satisfies the condition $(A_{\frac{1}{p}})$ for $p \in (0, 1)$ or the condition (b_q^-) with $q = \frac{p}{p-1}$ and $p \in (-\infty, 0)$, then the dual optimizer satisfies the same condition. In both cases, the weighted norm inequalities for the dual optimizer are further equivalent to the boundedness of the so-called opportunity process $L(p)$, the reduced form of the associated value process, see [Nut10]. The equivalences follow from the homogeneity of the power function, convex duality results and the dynamic programming principle. Recently, relying on convex duality arguments, [KW16] derived a sufficient condition for the dual optimizer $\widehat{Y}$ to satisfies a probabilistic Mckenhoupt's condition for U satisfying (1.15) with $a \in (0, 1)$. The critical case $a = 1$ which includes for example the logarithmic utility is addressed only for continuous S.

The first contribution of this chapter is a set of *a priori* estimates for L which show that the norms of L can be controlled by that of $L(p)$, where $p \in (-\infty, 0) \cup (0, 1)$ and depends only on a and b. As a result of the *a priori* estimates, we can benefit from the result of [Nut10] to derive sufficient conditions on S ensuring that L is uniformly bounded. We then proceed to translate the boundedness of L into integrability properties for the optimizers.

Our second contribution is the study of weighted norm inequalities for the dual optimizer $\widehat{Y}$ and the moments of the optimal wealth process $\widehat{X}$ via the process L. We begin by proving two results which give a complete picture of the weighted normed inequalities for $\widehat{Y}$ and the boundedness property of L. Our first result focuses on the case where U satisfies (1.15) with $a > 1$. For this case, we show the equivalence between the boundedness of L and reverse Hölder's inequality (b_q^-) for $\widehat{Y}$. Our equivalence generalizes the earlier result by [MT03b, Nut10] which focuses only on power utilities with negative risk aversion parameter. Our second result pertains to a Muckenhoupt's condition for U satisfying (1.15) without any restriction on a. Here, we show that the boundedness of L coupled with the existence of a dual supermartingale deflator satisfying Muckenhoupt's (A_k) for some arbitrary $k > 1$ ensure that the dual optimizer $\widehat{Y}$ satisfies (A_r) with $r = 1 + bk$. It turns out the boundedness property of L alone is not always sufficient to guarantee Muckenhoupt's condition for the dual optimizer, but serves as a necessary condition. In the particular case where $a = b \in (0, 1)$, we show that there is equivalence between the boundedness of L and Muckenhoupt's condition for the dual optimizer. The latter result generalizes the equivalence obtained for power utilities with positive risk aversion [Nut10] while the previous one extends the result by [KW16]. Next, we relate the boundedness of L to the moments of the optimal wealth process $\widehat{X}$. There is presently no indication why $\widehat{X}$ should be integrable w.r.t. the reference probability measure. Hence, we look for the moments of $\widehat{X}$ w.r.t. an equivalent local martingale measure (ELMM) $\widehat{\mathbb{Q}}$. We add two assumptions on the model, namely that the dual optimizer $\widehat{Y}$ is a local martingale and the stochastic logarithm of $\widehat{X}$ admits bounded jumps. Both assumptions are satisfied for continuous S. The local martingale property of $\widehat{Y}$ ensures that the probability measure $\widehat{\mathbb{Q}}$ with Radon-Nikodym derivative $d\widehat{\mathbb{Q}}/d\mathbb{P} = \widehat{Y}_T/\widehat{Y}_0$ is an equivalent local martingale measure. The condition on the jumps of $\widehat{X}$ will allow us to represent $\widehat{X}$ as the stochastic exponential of a BMO martingale. Our main result shows that if there exist $k > 1$ and a supermartigale deflator Z satisfying the condition (A_k), and L is uniformly bounded, then there exists $\gamma > 1$ such that the running maximum of $\widehat{X}$ at date T admits moments of order γ w.r.t. the measure $\widehat{\mathbb{Q}}$. Moreover, the stochastic integral of the optimal trading strategy w.r.t. the stochastic logarithm of the asset price process S is a BMO martingale w.r.t. $\widehat{\mathbb{Q}}$.

We exhibit two classes of market models for the assets , for which the boundedness of L can be easily verified. The first class is the class of exponential Lévy models for which we show that L is always bounded. Our second class is the class of continuous market models for which we show that the boundedness of L is reduced to a mild BMO condition on the market price of risk. For the latter models, the dual optimizer and optimal wealth are stochastic exponentials of local martingales w.r.t. the *minimal martingale measure* (see [Sch95]) and the underlying probability measure, respectively. With the previous results on weighted normed inequalities, we infer that $\widehat{X}$ and $\widehat{Y}$ are uniformly bounded in L^p for some $p > 1$. Moreover, the optimal trading strategy possesses a BMO property. Equipped with the knowledge of normed spaces for the optimizers, we obtain our third contribution. It addresses the stability analysis of the optimizers w.r.t. misspecification in the risk preference and initial capital in the setup of a continuous asset price model. We work under a BMO condition on the market price of risk ensuring the boundedness of the generalized opportunity process for utilities satisfying (1.15). For the optimal wealth process and dual optimizer, we establish a stability result in the Emery topology and the topology of uniform convergence on semimartingales. For the optimal trading strategy, we show stability in the L^2-topology. Our arguments rely on martingale inequalities and semimartingale convergence theorems to extend the result by [Kar10] from terminal time to the whole path. Stability of the optimal wealth and dual optimizer in the Emery topology have been obtained in the specific cases where the utility functions are of power type and misspecification is w.r.t. risk aversion parameter, see [Nut12c, MW13]. We note that results in [Nut12c, MW13] rely on stability results for BSDEs and the filtration is required to be continuous. To the best of our knowledge, our stability results in the semimartingale topology and topology of uniform convergence for general utilities have not yet appeared in the literature.

Chapter 3: FBSDEs for utility maximization: analysis of solution and risk aversion asymptotics

This chapter deals with the description of the solution to the utility maximization problem from terminal wealth. We work with three times continuously differentiable utility functions U defined on the positive real line. We also assume also that our asset price process S is the stochastic exponential of a local martingale R which has the canonical decomposition

$$dR = dM + d\langle M\rangle\mu,$$

M is an $\mathbb{R}^n$-valued continuous local martingale and μ the market price of risk. According to the previous chapter, the *generalized opportunity process* L is a nice tool to investigate the integrability properties of the optimizers: optimal trading strategy , optimal wealth process $\widehat{X}$ and dual optimizer $\widehat{Y}$. In this Chapter, we study the dynamical behavior of L, and then illustrate its importance in providing explicit formulas for the optimizers and in the study of their continuous behavior w.r.t. risk aversion.

We start by showing that L is a special semimartingale. This ensures that it has the canonical decomposition $L = L_0 + \int_0^\cdot Z^L dM + N^L + A^L$ where Z^L is a predictable process, N^L a local martingale orthogonal to M and A^L a predictable process of finite variation. Using the duality characterization of optimality of $\widehat{X}$ and $\widehat{Y}$, we show that the quadruple $(\widehat{X}, L, Z^L, N^L)$ can be identified as the unique solution to a system of fully coupled forward backward stochastic differential equations (FBSDEs) having a so-called *martingale property*. More precisely, for $t \in [0,T]$ we have

$$\begin{cases} d\widehat{X}_t &= x - \int_0^t \frac{U'(\widehat{X}_s)}{U''(\widehat{X}_s)}\left(\mu_s + \frac{Z_s^L}{L_{s-}}\right) dM_s - \int_0^t \frac{U'(\widehat{X}_s)}{U''(\widehat{X}_s)}\left(\mu_s + \frac{Z_s^L}{L_{s-}}\right)^\top d\langle M\rangle_s \mu_s, \\ L_t &= 1 - \int_t^T Z_s^L dM_s + N_t^L - N_T^L - \int_t^T L_{s-}\Phi_U(\widehat{X}_s)\left(\mu_s + \frac{Z_s^L}{L_{s-}}\right)^\top d\langle M\rangle_s \left(\mu_s + \frac{Z_s^L}{L_{s-}}\right), \end{cases} \tag{1.16}$$

with $\Phi_U(z) = 1 - \frac{1}{2}\frac{U^{(3)}(z)U'(z)}{|U''(z)|^2}, z > 0$. The concept of the solution with the *martingale property* is equivalent to a minimality property of L in a suitable class of processes. It is implicitly present in [HHI⁺14] and necessary for the *verification principle*. The minimality property of L was observed in the special case of power utility by [Nut12a]. The system (1.16) leads to explicit representations of the optimal trading strategy and the dual optimizer . This extends known results for classical utility functions [HIM05, Mor09a, FMW12] to more general ones.

The system (1.16) and the explicit formula for the optimizers offer the possibility to investigate their stability w.r.t. input data (e.g. market price of risk, risk aversion) using stability results from BSDEs theory [Mor09a, MW12]. However, such results require that L and $M^L = Z \cdot M^L + N^L$ lie in suitable normed spaces, e.g. that L is bounded and M^L is a BMO martingale. Unfortunately such properties of L and M^L were provided in [HHI⁺14, Section 4.2] only in the setting of a complete market and for a bounded market price of risk μ. In the context of an incomplete market, the normed spaces have been studied solely for power utilities [MT03b, HIM05, Mor09a, Nut12c, FMW12]. In the latter case, the normed spaces have proven effective for the stability analysis of the optimizers w.r.t. risk aversion parameter and market price of risk, see [MT03b, Nut12c, MW13]).

In this chapter, we provide two main results regarding the normed spaces for L and M^L. Under an exponential moments condition on the market price of risk and in the setting of a continuous filtration, our first result shows that $|\log L|$ admits exponential moment of all orders while M^L has moments of all orders. Our second result shows that L is uniformly bounded, and $M^L = Z^L \cdot M + N^L$ is a BMO martingale provided the market price of risk satisfies a certain BMO condition. The main difficulty in both results is to establish the integrability properties of L. We achieve this by exploiting *a priori* estimates of L derived in Chapter 2 which show that the norm of L can be controlled by that of the opportunity process for power utilities $L(p)$ and the knowledge of the normed spaces associated with $L(p)$ [MT03b, HIM05, Mor09a, Nut12c, FMW12]. The normed spaces of M^L are derive from those of L and usual arguments from BSDEs theory [MW12, Mor09b]. We note that the system of equations (1.16) is fully coupled, and we do not require the filtration to be continuous for our second result nor that the market of price of risk μ be bounded. A by-product of our result is therefore that it provides an example of system of FBSDE for which coefficients are unbounded, but the backward equation of the system nevertheless admits a bounded solution.

We then employ BSDEs stability theorems to study the robustness of the optimizers w.r.t. relative risk aversion. To this end, we consider a sequence of utility functions $(U_m)_{m\in\mathbb{N}}$ with sequence of relative risk aversions converging uniformly to $c = 1$ resp. $c = +\infty$. We show that the resulting limit of the optimal wealth processes $(\widehat{X}^m)_{m\in\mathbb{N}}$, optimal trading strategies $(\nu^m)_{m\in\mathbb{N}}$ and dual optimizers $(\widehat{Y}^m)_{m\in\mathbb{N}}$ are related to those of the utility maximization problem with logarithmic resp. exponential utility for $c = 1$ resp. $c = +\infty$. If the market price of risk possesses exponential moments of all orders, we obtain the convergence of $(\widehat{X}^m)_{m\in\mathbb{N}}$ and $(\widehat{Y}^m)_{m\in\mathbb{N}}$ in the Emery topology, and the convergence of $(\nu^m)_{m\in\mathbb{N}}$ in the L^2-topology. Our results generalize those obtained in [MW13, Nut12c] for $U_m(z) = \frac{z^{p_m}}{p_m}, z > 0$ and $(p_m)_{m\in\mathbb{N}}$ a sequence in $(-\infty, 0) \cup (0, 1)$ with $1 - p_m \to c$. Under a BMO condition on the market price of risk, we obtain convergence of the sequence of *dual optimal martingale measures* $(\widehat{\mathbb{Q}}^m)_{m\in\mathbb{N}}$ with $d\widehat{\mathbb{Q}}^m/d\mathbb{P} = \widehat{Y}^m_T/\widehat{Y}^m_0, m \in \mathbb{N}$ to an equivalent local martingale measure $\mathbb{Q}$. The convergence is in *relative entropy*, i.e.,

$$\lim_{m\to+\infty} \mathbb{E}\left[\frac{d\widehat{\mathbb{Q}}^m}{d\mathbb{P}} \log \frac{d\widehat{\mathbb{Q}}^m}{d\mathbb{Q}}\right] = 0.$$

In the case $c = 1$, $\mathbb{Q}$ corresponds to the *minimal martingale measure* while for $c = +\infty$ it is the *minimal entropy martingale measure* (see [Fri00, GR02]). This convergence has already been obtained in [MT03b] for the particular case $U_m(z) = \frac{z^{p_m}}{p_m}, z > 0$. Let us mention that the case

$c = +\infty$ has been treated in [GT11] in the context of a complete Black-Scholes model with bounded risk aversion using convex duality and PDE methods. For an earlier result in a discrete financial model, we refer to [CR06].

Chapter 4: Optimal stopping problems in a progressively enlarged filtration: a two step decomposition approach

This chapter deals with the description of solutions to optimal stopping problems in a progressively enlarged filtration. Let τ be a random time and D the default indicator process of τ, i.e.

$$D_t = 1_{\{\tau \le t\}},\ t \in [0, T].$$

The underlying filtration $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ is given by the progressive enlargement of a reference filtration $\mathbb{F}$ by τ. We assume the density hypothesis on the conditional distribution of τ given $\mathbb{F}$. This ensures that every $\mathbb{G}$-optional process satisfies the *optional splitting formula*, i.e. it can be identified with an $\mathbb{F}$-optional process before τ and a $\mathbb{G}^\tau$-optional process after τ. Here $\mathbb{G}^\tau = (\mathcal{G}^\tau_t)_{t\ge 0}$ is the initial enlargement of $\mathbb{F}$ by τ. A result by [EI18] shows that every $\mathbb{G}^\tau$-stopping time is of the form $\gamma^d(\tau)$ where $\gamma^d(\cdot)$ is a stopping time w.r.t. the product filtration $(\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+))$. [BZ14] has shown that for every $\mathbb{G}$-stopping time γ, there exists an $\mathbb{F}$-stopping time γ^{b} and a $\mathbb{G}^\tau$-stopping time $\gamma^d(\tau)$ such that $\gamma^d(\tau) \ge \tau$ and $\gamma = \gamma^{\mathrm{b}} 1_{\{\gamma^{\mathrm{b}} < \tau\}} + \gamma^d(\tau) 1_{\{\gamma^{\mathrm{b}} \ge \tau\}}$. In this context, we call γ^{b} the pre-default value of γ and $\gamma^d(\tau)$ as the post-default value of γ.

For the optimal stopping problem, we consider as reward a $\mathbb{G}$-adapted càdlàg process ζ. We assume that ζ sufficiently integrable and that there exists an $\mathcal{O}(\mathbb{F})$-measurable process ζ^{b} and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process ζ^d such that ζ satisfies the *optional splitting formula*

$$\zeta_t \quad = \quad \zeta^{\mathrm{b}}_t 1_{\{t<\tau\}} + \zeta^d_t(\tau) 1_{\{t \ge \tau\}},\ t \in [0, T]. \tag{1.17}$$

We develop a two-step algorithm to solve the optimal stopping problem with time horizon $T > 0$ and reward ζ in the filtration $\mathbb{G}$ leading to the precise description of the pre-default and post-default values of optimal stopping times.

Following [JP11, Pha10, JKP13] on the utility maximization problem with asset price subject to the default event modeled by τ, a natural approach to solve the optimal stopping problem with reward ζ is to split the stopping problem into two sub-problems of optimal stopping: an optimal stopping problem in the filtration $\mathbb{F}$ whose solutions provide the pre-default values of solutions to the original problem, and another stopping problem in the filtration $\mathbb{G}^\tau$ whose solutions provide post-default values of solutions to the original problem. The Snell envelope V of ζ being the main tool to characterize optimality, our method to identify the sub-problems of optimal stopping is to determine the explicit expressions of the pre-default resp. post-default values V^{b} resp. $V^d(\tau)$ of the Snell envelope and to link these values respectively to the rewards of the sub-problems of optimal stopping. We show that the reward of the sub-problem of optimal stopping in the filtration $\mathbb{G}^\tau$ is given by $\zeta^d(\tau)D$ and its Snell envelope by $V^d(\tau)$. For the sub-problem of optimal stopping in the filtration $\mathbb{F}$, we show that the reward Υ is given by

$$\Upsilon = \zeta^{\mathrm{b}} G + \int_0^{\cdot} V^d_u(u) \alpha^d_u(u) \eta(du),$$

where $G_t = \mathbb{P}(\tau > t | \mathcal{F}_t)$ is the *conditional survival process* of τ, η the law of τ and $(\alpha^d_t(u))_{t\ge 0}$ the conditional density process of τ given $\mathbb{F}$. Moreover, the Snell envelope of Υ is given by the process $V^{\mathrm{b}} G + \int_0^{\cdot} V^d_u(u) \alpha^d_u(u) \eta(du)$.

Our main result shows that a $\mathbb{G}$-stopping time $\overline{\nu} = \overline{\nu}^{\mathrm{b}} 1_{\{\overline{\nu}^{\mathrm{b}} < \tau\}} + \overline{\nu}^d(\tau) 1_{\{\overline{\nu}^{\mathrm{b}} \ge \tau\}}$ is an optimal stopping time for our original stopping problem if and only if $\overline{\nu}^{\mathrm{b}}$ is an optimal stopping time for the optimal stopping problem with reward Υ, and $\overline{\nu}^d(\tau)$ is an optimal stopping time for the stopping problem with reward $\zeta^d(\tau)D$ at time $T \wedge \tau$. The dependence of the reward Υ on the process $(V^d_u(u))_{u\in[0,T]}$ suggests the following algorithm to construct the pre-default and post-default values of optimal stopping times. First solve the optimal stopping problem with reward $\zeta^d(\tau)D$ to obtain the post-default value of an optimal stopping time and the Snell envelope $V^d(\tau)$. Then solve the optimal stopping problem with reward Υ to obtain the pre-default value of an optimal stopping time. A key benefit of the algorithm is the insight its provides on the actions an agent will have to take in presence of default risk both before and after the default event in order to solve the optimal stopping problem. We illustrate the importance of our two step algorithm by determining explicit hedges for defaultable claims of American type before and after default in a complete market consisting of a default free asset and a defaultable zero coupon bond.

Another contribution of this chapter is a recursive formula for the pre-default value V^{b} of the Snell envelope V. As V is the Snell envelope of ζ, its dynamics is described by the RBSDE with data $(0, \zeta, \zeta_T)$. Intuitively, the dynamics of V^{b} should be described by a RBSDE in the filtration $\mathbb{F}$. However, the identification of $V^{\mathrm{b}}G + \int_0^{\cdot} V^d_u(u)\alpha^d_u(u)\eta(du)$ as the Snell envelope of Υ is not very convenient to determine the data of the corresponding RBSDE. We obtain an alternative representation of V^{b} from which one can derive the parameters of the RBSDE describing the dynamics of V^{b}.

Chapter 5: Reflected BSDEs in a progressively enlarged filtration: a two step decomposition approach

In this chapter, we investigate the existence of solutions to reflected BSDEs in an enlarged filtration $\mathbb{G}$. We assume that $\mathbb{G}$ is given by the *progressive enlargement* of a reference filtration $\mathbb{F}$ with a random time τ. We assume the density hypothesis on the regular conditional distribution of τ given $\mathbb{F}$ and that the reference filtration $\mathbb{F}$ supports an $\mathbb{R}^n$-valued Brownian motion B. We denote by $\mathbb{G}^\tau$ the initial enlargement of $\mathbb{F}$ by τ. We recall that a reflected BSDE in the filtration $\mathbb{G}$ with driver F, obstacle process S and terminal value ξ has the form

$$\begin{cases} Y_t = \xi + \int_t^T F(s, Y_{s-}, Z_s, U_s)ds + K_T - K_t - \int_t^t Z_s dB_s^{\mathbb{G}} - \int_t^T U_s dN_s^{\mathbb{G}} - M_T + M_t, \\ Y_t \geq S_t, \\ \int_0^T (Y_{s-} - S_{s-})dK_s = 0,\ t \in [0,T]. \end{cases} \tag{1.18}$$

In Chapter 5 we focus on existence of solutions to the RBSDE with data (F, S, ξ) and the precise description of the pre-default and post-default values of the solutions. For investigating existence of solutions, we identify two alternative weakly coupled systems of RBSDEs whose solutions lead to the existence of a solution (Y, Z, U, M, K) to our original RBSDE. The first one is an RBSDE in the filtration $\mathbb{G}^\tau$ whose solution $(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau))$ leads to the post-default value of (Y, Z, U, M, K) , and the second one a RBSDE in the filtration $\mathbb{F}$ whose solution $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, M^{\mathrm{b}}, K^{\mathrm{b}})$ leads to the pre-default value of (Y, Z, U, M, K). The RBSDE in $\mathbb{F}$ is w.r.t. an auxiliary probability measure equivalent to the reference measure, and its solution depends on the solution of the first RBSDE in the filtration $\mathbb{G}^\tau$. The solution (Y, Z, U, M, K) is then obtained by suitably pasting the solutions resulting from the alternative RBSDEs, namely by

$$\begin{cases} Y_t &= Y_t^{\mathrm{b}} 1_{\{t<\tau\}} + Y_t^{\mathrm{d}}(\tau) 1_{\{t\geq\tau\}}, \\ Z_t &= Z_t^{\mathrm{b}} 1_{\{t\leq\tau\}} + Z_t^{\mathrm{d}}(\tau) 1_{\{t>\tau\}}, \\ U_t &= (Y_t^{\mathrm{d}}(t) - Y_{t-}^{\mathrm{b}}) 1_{\{t\leq\tau\}}, \\ M_t &= M_{t\wedge\tau}^{\mathrm{b}} + (M_t^{\mathrm{d}}(\tau) - M_\tau^{\mathrm{d}}(\tau)) 1_{\{t\geq\tau\}}, \\ K_t &= K_{t\wedge\tau}^{\mathrm{b}} + (K_t^{\mathrm{d}}(\tau) - K_\tau^{\mathrm{d}}(\tau)) 1_{\{t>\tau\}},\ t \in [0,T]. \end{cases}$$

To identify the alternative RBSDEs, we make use of the one-to-one correspondence between solutions to RBSDEs and their Snell envelope representation [EKKP+97, LX05, Kli15], and the optional splitting formula for Snell envelopes established in Chapter 4. A delicate issue to address in our approach is to justify the pasting procedure, since we deal with different filtrations, and stochastic integration is not stable under a change of filtration [Jeu80, CMS80, Jac80, Wei84]. We overcome this obstacle by expressing stochastic integrals w.r.t. $\mathbb{G}$-local martingales as stochastic integral w.r.t. $\mathbb{F}$-local martingales and $\mathbb{G}^\tau$-local martingales.

Having reduced the existence of a solution to (1.18) to that of two alternative RBSDEs, we then proceed to examine cases in which solutions exist. We first consider the case where F is globally Lipschitz, the data (F, S, ξ) belong to $\mathbb{L}^p$ for some $p > 1$ and $\mathbb{F}$ is a general filtration. In this case, we show that a solution exists. Our result extends the one in [BPTZ15] which deals with a driver F depending only on (Y, Z). Let us mention that the result of [BPTZ15] does not lead to the description of the pre-default and post-default values of the solution, since the contraction principle is employed in the filtration $\mathbb{G}$ directly. We then consider the case where F has linear growth in Y, quadratic growth in Z and is locally Lipschitz continuous in U. In this case, we assume the filtration $\mathbb{F}$ to be the augmented filtration of the Brownian motion B, and the obstacle S to be bounded and as well as the terminal value ξ. Under the latter assumptions, we prove the existence of a solution with bounded value process. Results obtained so far for RBSDEs with quadratic growth have been obtained only for continuous filtrations [KLQT02, Lio14]. Under the additional assumptions that F is globally Lipschitz continuous in Y and locally Lipschitz continuous in (Z, U), we establish *a priori* estimates of the solutions for two distinct data in suitably normed spaces. Based on the *a priori* estimates we show the uniqueness of a bounded solution. This uniqueness result entails that every bounded solution is given by our pasting procedure. We then exploit our uniqueness result and optional splitting formula for solutions to establish a comparison principle for RBSDEs in the filtration $\mathbb{G}$. A key feature of our comparison principle is that it requires a weaker version of a sufficient condition usually employed in the literature of BSDEs with jumps to employ a measure change when proving comparison principle, see [Roy06].

2. Utility maximization problem: integrability properties of the solution and stability analysis

2.1 Introduction

In this chapter, we consider the problem of maximization of utility from terminal wealth for an agent investing in the financial market with asset prices modeled by a semimartingale S. The question of existence of an optimal trading strategy has been treated using methods of convex duality, see [KS99, KS03] and references therein. However, a description of optimal trading strategies that is amenable to numerical computations, and their stability analysis w.r.t. misspecification in the utility functions, initial capital or some parameters of the assets have been mainly studied for logarithmic and power utilities. Our goal in this chapter is to lay out a foundation that will enable us to describe optimal trading strategies and study their stability analysis for fairly general utility functions U defined on the positive real line and continuous S.

For the logarithmic utility function $U(z) = \log z$, optimal trading strategies are known to depend solely on the semimartingale differential characteristics of the assets and have been determined by [GK00, Kal00]. Regarding their stability analysis w.r.t. market price of risk and initial capital, we refer to [Kar10, MW13]. For the class of power utilities $U(z) = \frac{z^p}{p}$ where $p \in (-\infty, 0) \cup (0, 1)$, the homogeneity feature of the power functions and the dynamic programming principle reduce the description of optimal trading strategies and their stability analysis to the knowledge of the so-called *opportunity process* $L(p)$, see [Nut10, Nut12a, Nut12c]. The opportunity process $L(p)$ describes for $t \in [0, T]$ the maximal conditional expected utility that can be achieved on $[t, T]$ with unit capital. The dynamical behavior of $L(p)$ is described by a backward stochastic differential equation (BSDE) and explicit formulas of optimal trading strategies are given in terms of $L(p)$, see [MT03b, Nut12a, MT03a, HIM05]. Properties of the norms of $L(p)$ such as the boundedness property or the existence of moments of a certain order have been well studied, see [MT03b, Nut10, Nut12c, FMW12, MW13]. Building on the knowledge of the integrability properties of $L(p)$, arguments from the theory of BSDEs and the formulas for optimal trading strategies in terms of $L(p)$, stability results for optimal trading strategies w.r.t. market price of risk and risk aversion parameter p have been obtained in [MT03b, Nut12a, MW13].

For U sufficiently smooth and continuous S, [HHI$^+$14, ST14] employed the stochastic maximum principle of [Pen93] to derive a system of forward-backward stochastic differential equations (FBSDEs) describing the joint dynamics of $(\widehat{X}, L)$ where $\widehat{X}$ is the optimal wealth, and the process L is given by

$$L = \frac{\widehat{Y}}{U'(\widehat{X})}$$

with the solution $\widehat{Y}$ to the dual problem from convex duality. They papers provide an explicit formula for the optimal trading strategy in terms of L. The formula for the optimal trading strategy sheds light on its dependence on the market price of risk, risk aversion and initial capital. In view of the description of the optimal trading strategy, one could study its stability using L and stability arguments from the theory of BSDEs provided L lies in suitable normed spaces, e.g. that L is bounded or has moments of certain order. Unfortunately, normed spaces for L are not known except in the particular case of complete market models with bounded market price of risk or the cases of logarithmic and power utilities. For logarithmic utility, L is given by the constant 1 while for power utility with power p, L coincides with the opportunity process $L(p)$, see [Nut10]. For this reason, we will call L the generalized opportunity process.

In the framework of general semimartingales, in this Chapter we provide a set of *a priori* estimates for L from which we derive necessary and sufficient conditions for its boundedness and integrability properties of the optimizers: weighted norm inequalities for the dual optimizer $\widehat{Y}$ such as the reverse Hölder inequality for some $q \in (0,1)$ or the probabilistic Muckenhoupt condition (A_r) for some $r > 1$ (see Definition 2.2.4) , moments of the running maximum of the optimal wealth $\widehat{X}$ and a BMO property for the optimal trading strategy. The *a priori* estimates also set the stage for the study of solutions to the system of FBSDEs describing the joint dynamics of $(\widehat{X}, L)$, and the investigation of the asymptotic behavior of the optimizers w.r.t. relative risk aversion carried out in Chapter 3. We build on the integrability properties of the optimizers to study their stability w.r.t. misspecification in the utility function and initial capital for continuous S in various topologies. The study of the integrability properties of the optimizers is also motivated by some practical applications. Indeed, weighted normed inequalities guarantee that the dual optimizer is of class (D), and thus when possessing the local martingale property, the probability measure with Radon Nikodym density process $\widehat{Y}/\widehat{Y}_0$ defines a pricing measure known as the *dual optimal martingale measure*, see [KW16]. The existence of the dual optimal martingale measure leads to the uniqueness and simple formulas for marginal utility-based pricing for bounded contingent claims and asymptotic utility-based hedging strategies, see [HKS05, KS07]. In the setup of incomplete markets where several pricing measures exist for the valuation of contingent claims, the dual optimal martingale measure is a suitable choice as it takes into account the risk preference of the investor and his initial capital. Unfortunately the dual optimizer may fail to be of class (D), see [KS99, Example 5.1]. The integrability properties of the optimizers lead to a reduction of the domain of the primal and dual problems. The reduction of the domain of the optimization problems might be of interest for numerical algorithms based on the artificial market completion to simulate $\widehat{X}$ and $\widehat{Y}$, see [BHM13].

The rest of the chapter is structured as follows. In Section 2.2, we fix notation and discuss some basic concepts regarding weighted norm inequalities. In addition, we state the utility maximization problem and recall some results from the convex duality approach of [KS99]. Section 2.3 introduces the generalized opportunity process L. The *a priori* estimates for L which lead to necessary and sufficient conditions for the boundedness property of L are obtained in Section 2.4. In section 2.5, we prove results linking the boundedness property of L to the integrability properties of the dual optimizers. Section 2.6 presents market models for which our conditions for the boundedness property of L are satisfied. Our stability result is given in Section 2.7. In Section 2.8, we collect some results on BMO martingales necessary for our proofs.

2.2 Preliminaries

Let $T \in (0, +\infty)$ be a fixed time horizon. Throughout this chapter, we work with a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and completeness, and $\mathcal{F}_0$ being trivial. $\mathcal{T}(\mathbb{F})$ is the set of stopping times with values in $[0,T]$ and for $\tau \in \mathcal{T}(\mathbb{F})$, we denote by $\mathcal{F}_\tau$ the sub-sigma algebra of $\mathcal{F}$ generated by real valued $\mathbb{F}$-adapted càdlàg processes stopped at τ. For a given probability measure $\mathbb{Q}$, $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation w.r.t. the measure $\mathbb{Q}$. For $\mathbb{Q} = \mathbb{P}$, we will simply write $\mathbb{E}$. Let $n \in \mathbb{N}$. For $z \in \mathbb{R}^n$, we denote by $z^\top$ its transpose and by $||z|| = \left(z^\top z\right)^{\frac{1}{2}}$ its Euclidean norm. For an $\mathbb{R}^n$-valued semimartingale N and an $\mathbb{R}^n$-valued predictable integrand π, the stochastic integral, denoted by $\int_0^\cdot \pi dN$ or $\pi \cdot N$ is the scalar semimartingale with initial value zero given by $\int_0^\cdot \pi dN = \sum_{i=1}^n \int_0^\cdot \pi^i dN^i$. We denote by $\mathcal{L}(N)$ the set of $\mathbb{R}^n$-valued predictable integrands π, for which $\int_0^\cdot \pi dN$ is well defined. For two real-valued semimartingales N, M, we denote by

$[N, M]$ the quadratic covariation of N and M. The predictable quadratic covariation of two locally square integrable martingales N, M is denoted by $\langle N, M\rangle$. We will simply write $\langle N\rangle$ or $[N]$ if $N = M$. We recall that a real-valued local martingale L is orthogonal to an $\mathbb{R}^n$-valued local martingale $M = (M^1, \cdots, M^n)$ if and only if $[L, M^i]$ is a local martingale for $i = 1, \cdots, n$.

We introduce some spaces that will play an important role in the sequel. Let $r \geq 1$ and $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{A})$. $L^r(\mathbb{Q})$ (resp. $L^\infty(\mathbb{Q})$) is the space of $\mathcal{F}_T$-measurable real valued random variables H such that $\mathbb{E}^{\mathbb{Q}}[|H|^r] < +\infty$ (resp. $||H||_\infty = \operatorname{ess\,sup}_{\omega\in\Omega}|H(\omega)| < +\infty$). We denote by $\mathcal{S}^r(\mathbb{Q})$ (resp. $\mathcal{S}^\infty(\mathbb{Q})$) the space of real valued càdlàg semimartingales Z such that $||Z||_{\mathcal{S}^r(\mathbb{Q})} = \left(\mathbb{E}\left[\sup_{t\in[0,T]}|Z_t|^r\right]\right)^{\frac{1}{r}} < +\infty$ (resp. $||Z||_\infty = \left|\left|\sup_{t\in[0,T]}|Z_t|\right|\right|_\infty < +\infty$). We simply write $\mathcal{S}^r$ (resp. $\mathcal{S}^\infty$) if $\mathbb{P} = \mathbb{Q}$. A real valued càdlàg semimartingale Z is bounded if $Z \in \mathcal{S}^\infty$. For a real valued càdlàg semimartingale Z, Z_- denotes the process of left limits, i.e. $Z_{t-} = \lim_{s\nearrow t} Z_s$ $(Z_{0-} = Z_0)$. We denote the jump process of Z by ΔZ.

Throughout this work, equalities and inequalities between random variables are understood in the almost sure sense while equalities and inequalities between random processes are up to indistinguishability unless mentioned otherwise. The Doleans-Dade exponential of a semimartingale Z will be denoted by $\mathcal{E}(Z)$. For two stopping times $\sigma, \tau \in \mathcal{T}(\mathbb{F})$ with $\sigma \leq \tau, \mathcal{E}(Z)_{\sigma,\tau} = \mathcal{E}(Z)_\tau/\mathcal{E}(Z)_\sigma$.

2.2.1 Some crucial concepts

We begin by recalling some terminology that will be often used.

Definition 2.2.1.

- *A filtration $\mathbb{G}$ is said to be* continuous *if all $\mathbb{G}$-local martingales have continuous paths.*
- *A real valued adapted càdlàg process Z is of class (D) if the family $\{Z_\sigma, \sigma \in \mathcal{T}(\mathbb{F})\}$ is uniformly integrable.*
- *A positive real valued adapted process Z is bounded away from 0 (resp. ∞) if there exists a constant $c > 0$ (resp. $\delta > 0$) such that for all $t \in [0, T], Z_t \geq c$ (resp. $Z_t \leq \delta$).*

BMO martingales

We recall the notion of BMO martingales. Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{A})$.

Definition 2.2.2. *A $\mathbb{Q}$-local martingale N with $N_0 = 0$ is a BMO martingale if and only if there exists a constant $\alpha > 0$, such that for every $\tau \in \mathcal{T}(\mathbb{F})$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[[N]_T - [N]_{\tau_-}|\mathcal{F}_\tau\right] \leq \alpha^2. \tag{2.1}$$

The smallest constant α for which (2.1) holds is the BMO-norm of N. We denote it by $||N||_{\mathrm{BMO}(\mathbb{Q})}$. We refer to $BMO(\mathbb{Q})$ as the class of $\mathbb{Q}$-local martingales N with $N_0 = 0$ and for which $||N||_{\mathrm{BMO}(\mathbb{Q})} < +\infty$. For an introduction to BMO martingales, we refer to [DDM79, Kaz94].

Definition 2.2.3. *Let Z be a strictly positive càdlàg semimartingale with $Z_- > 0$.*

1. *Let $\gamma \in \mathbb{R}\backslash\{0\}$. We say that Z satisfies the condition (b_γ) if there exists a constant $C_1 \geq 1$ such that for all stopping times $\sigma \in \mathcal{T}(\mathbb{F})$, we have*

$$\frac{1}{C_1} Z_\sigma \leq \left(\mathbb{E}\left[Z_T^\gamma|\mathcal{F}_\sigma\right]\right)^{\frac{1}{\gamma}} \leq C_1 Z_\sigma. \tag{2.2}$$

Z satisfies $\left(b_\gamma^-\right)$ (resp. $\left(b_\gamma^+\right)$) if solely the left hand (resp. right) side of (2.2) holds.

2. *Z satisfies the condition (J) if there exists a constant $C_2 > 0$ such that*

$$\frac{1}{C_2} Z_- \quad \leq Z \leq \quad C_2 Z_- .$$

The conditions (b_γ) and (J) were introduced in [DDM79] for the study of the stochastic exponential of BMO martingales. The condition (b_γ) encompasses the more familiar (A_r) and (R_r) conditions which we now state.

Definition 2.2.4. *Let Z be a strictly positive càdlàg semimartingale and $r > 1$.*

- *Z satisfies the* Muckenhoupt condition *denoted by (A_r) if and only if there exists $C_1 > 0$ such that for all stopping times $\sigma \in \mathcal{T}(\mathbb{F})$ we have*

$$\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{1}{r-1}} \middle| \mathcal{F}_\sigma\right] \leq C_1.$$

- *Z satisfies the* reverse Hölder inequality *denoted by (R_r) if and only if there exists $C_2 > 0$ such that for all stopping times $\sigma \in \mathcal{T}(\mathbb{F})$ we have*

$$\mathbb{E}\left[\left(\frac{Z_T}{Z_\sigma}\right)^{r} \middle| \mathcal{F}_\sigma\right] \leq C_2.$$

Remark 2.2.5.
- *It is clear that for $\gamma < 0$, we have $\left(b_\gamma^-\right)$ is equivalent to (A_r) with $r = 1 - \frac{1}{\gamma}$ while for $\gamma > 1$, we have $\left(b_\gamma^+\right)$ is equivalent to (R_γ).*
- *Let $r > 1$. From Hölder's inequality, (A_r) implies $(A_{r'})$ for $r' \geq r$. Similarly, (R_r) implies $(R_{r'})$ for $r' \leq r$.*

The following lemma provides a link between (A_r) and $\left(b_q^-\right)$. It also shows that (A_r) and $\left(b_q^-\right)$ are sufficient conditions to guarantee the class (D) property of a process Z.

Lemma 2.2.6. *Let Z be a strictly positive càdlàg semimartingale and $r > 1$.*

i) Assume that Z satisfies (A_r). Then Z satisfies $\left(b_q^-\right)$ for any $q \in (0,1)$.

ii) Assume that Z is a supermartingale. If Z satisfies $\left(b_q^-\right)$ for some $q \in (0,1)$ then it satisfies $\left(b_l^-\right)$ for any $l \in (0,1)$.

iii) Assume that $\mathbb{E}[Z_T] < \infty$. If Z satisfies $\left(b_q^-\right)$ for $q \in (0,1)$, then Z is of class (D) .

Proof. i) Z satisfies (A_r). Thus there exists $C > 0$ such that $\mathbb{E}\left[\left(\frac{Z_\tau}{Z_T}\right)^{\frac{1}{r-1}} \middle| \mathcal{F}_\tau\right] \leq C$ for all $\tau \in \mathcal{T}(\mathbb{F})$. Let $q \in (0,1)$. Let $\alpha = \frac{q}{r}$, $\bar{r} = \frac{r}{r-1}$ and $\sigma \in \mathcal{T}(\mathbb{F})$. By Hölder's inequality we have

$$1 \quad = \quad \mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\alpha}\left(\frac{Z_T}{Z_\sigma}\right)^{\alpha} \middle| \mathcal{F}_\sigma\right] \leq \left(\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\alpha\bar{r}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{\bar{r}}}\left(\mathbb{E}\left[\left(\frac{Z_T}{Z_\sigma}\right)^{\alpha r} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{r}}.$$

Note that $\alpha\bar{r} = \frac{q}{r-1}$ and $\alpha r = q$. We infer from the above inequality that

$$\left(\mathbb{E}\left[\left(\frac{Z_T}{Z_\sigma}\right)^{q} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{r}} \quad \geq \quad \left(\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{q}{r-1}} \middle| \mathcal{F}_\sigma\right]\right)^{-\frac{r-1}{r}} .$$

The map $(0,+\infty) \ni x \mapsto x^q$ is concave. Thus applying Jensen's inequality and the fact that the function $(0,+\infty) \ni x \mapsto x^{-\frac{r-1}{r}}$ is decreasing, we obtain

$$\left(\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{q}{r-1}}\middle|\mathcal{F}_\sigma\right]\right)^{-\frac{r-1}{r}} \geq \left(\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right]\right)^{-q\frac{r-1}{r}} \geq C^{-q\frac{r-1}{r}}.$$

The above two inequalities lead to $\mathbb{E}\left[\left(\frac{Z_T}{Z_\sigma}\right)^q\middle|\mathcal{F}_\sigma\right] \geq C^{-\frac{r-1}{r}}$. Hence Z satisfies $\left(b_q^-\right)$.
ii) See [Nut10, Lemma 4.9].
iii) Assume that Z satisfies (b_q^-) for some $q \in (0,1)$. Then there exists $K_q > 0$ such that for every $\sigma \in \mathcal{T}(\mathbb{F})$ we have

$$\mathbb{E}\left[\left(\frac{Z_T}{Z_\sigma}\right)^q |\mathcal{F}_\sigma\right] \geq K_q.$$

Let $\sigma \in \mathcal{T}(\mathbb{F})$. By Jensen's inequality, we have $Z_\sigma^q K_q \leq \mathbb{E}\left[Z_T^q|\mathcal{F}_\sigma\right] \leq \left(\mathbb{E}\left[Z_T|\mathcal{F}_\sigma\right]\right)^q$. Hence

$$Z_\sigma \leq K_q^{-\frac{1}{q}}\mathbb{E}\left[Z_T|\mathcal{F}_\sigma\right], \ \sigma \in \mathcal{T}(\mathbb{F}).$$

Since $\mathbb{E}\left[Z_T\right] < \infty$, we deduce that the family $\{Z_\sigma, \ \sigma \in \mathcal{T}(\mathbb{F})\}$ is uniformly integrable. $\square$

The following proposition gives a useful characterization of the stochastic exponential of BMO martingales with jumps bounded from below.

Proposition 2.2.7. *[DDM79, Propositions 5 and 6] Let N be a $\mathbb{F}$-local martingale with $N_0 = 0$. Suppose that $\Gamma = \mathcal{E}(N)$ is a uniformly integrable martingale. The following assertions are equivalent:*

i) N is in BMO and there exists a constant $h > 0$ such that $1 + \Delta N \geq h$.

ii) Γ satisfies (J) and (A_r) for some $r \in (1,+\infty)$.

iii) Γ satisfies (J) and (R_k) for some $k \in (1,+\infty)$.

Proposition 2.2.7 admits the following corollary which shows the L^p-boundedness of the stochastic exponential of BMO-martingales.

Corollary 2.2.8. *Let $K_{BMO} > 0$ and $\epsilon > 0$. There exists $p > 1$ and $S_p > 0$ depending only on K_{BMO} and ϵ such that for every local martingale M satisfying $||M||_{BMO}^2 \leq K_{BMO}$ and $\epsilon \leq \Delta M + 1 \leq \frac{1}{\epsilon}$, the process $Z = \mathcal{E}(M)$ satisfies*

$$\sup_{t\in[0,T]} \mathbb{E}\left[Z_t^p\right] \leq S_p.$$

In particular $Z \in \mathcal{S}^p$.

The key feature of Corollary 2.2.8 is the dependence of p on the constants K_{BMO} and ϵ. We will rely on this dependence in Sections 2.6.2 and 2.7 to study the convergence of a sequence of stochastic exponential of BMO-martingales in the topology of uniform convergence. Corollary 2.2.8 appears in comments following Proposition 6 in [DDM79]. We will give a proof of Corollary 2.2.8 in Section 2.8 due to its importance in this chapter.

Semimartingale topology

We recall some results regarding the convergence of semimartingales. Let X, Y be two semimartingales. The Emery distance between X and Y is defined as follows :

$$d(X,Y) \quad := \quad |X_0 - Y_0| + \sup_{|H|\leq 1} \mathbb{E}\left[\sup_{t\in[0,T]} 1 \wedge |H \cdot (X-Y)_t|\right] \tag{2.3}$$

where the supremum is taken over all real valued predictable processes H bounded by one. The above distance induces on the space of semimartingales $\mathcal{S}$ a topology known as the *Emery topology* or the *semimartingale topology* (see [Eme79]). We will denote this topology by $\mathcal{S}_0$. Let X be a special semimartingale with canonical decomposition $X = X_0 + M^X + A^X$ where M^X and A^X are respectively the local martingale part and the predictable finite variation part. For $r \geq 1$, we define $||X||_{\mathcal{R}^r}$ as follows

$$||X||_{\mathcal{R}^r} \quad := \quad |X_0| + \left|\left|\int_0^T |dA^X|\right|\right|_{L^r} + \left|\left|\left[M^X\right]_T^{\frac{1}{2}}\right|\right|_{L^r},$$

The following proposition provides sufficient conditions for a sequence $(X^m)_{m\in\mathbb{N}}$ to converge in $\mathcal{S}_0$.

Proposition 2.2.9. *Let $r \geq 1$. Let $(X^m)_{m\in\mathbb{N}}$ be a sequence of càdlàg semimartingales and X a càdlàg semimartingale. The following assertions are true :*

a) If $(X^m - X)_{m\in\mathbb{N}}$ converges to 0 in the norm $||\cdot||_{\mathcal{R}^r}$, then $(X^m - X)_{m\in\mathbb{N}}$ converges to 0 in $\mathcal{S}_0$ (see [Eme79, Theorem 2]),

b) Assume for every $m \in \mathbb{N}$, X^m has continuous paths and X has continuous paths as well. Then $(\mathcal{E}(X^m))_{m\in\mathbb{N}}$ converges to $\mathcal{E}(X)$ in $\mathcal{S}_0$ if $(X^m)_{m\in\mathbb{N}}$ converges to X in $\mathcal{S}_0$ (see [Nut12c, Lemma A.2]).

2.2.2 The optimization problem

Let $R = (R^i)_{1\leq i\leq n}$ be an $\mathbb{R}^n$-valued semimartingale with $R_0 = 0$. We consider a financial market which consists of $n+1$ assets, i.e. n assets with one bond. The bond is assumed to be constant and equal to 1. The joint price process for the n assets is modeled by a strictly positive $\mathbb{R}^n$-valued semimartingale $S = (S^i)_{1\leq i\leq n}$ with $S^i = S_0^i\mathcal{E}(R^i), i = 1, 2\cdots, n$. We recall that a probability measure $\mathbb{Q} \sim \mathbb{P}$ is called an equivalent local martingale measure (hereafter ELMM) for S if S is a local martingale under $\mathbb{Q}$. We denote by $\mathcal{M}^e(S)$ the set of all such measures. In order to exclude a suitable notion of arbitrage opportunities, namely *free lunch with vanishing risk* (see [DS94]) we assume throughout this chapter that

Assumption 2.2.10. *$\mathcal{M}^e(S) \neq \emptyset$.*

A trading strategy is a predictable R-integrable $\mathbb{R}^n$-valued process $\pi = (\pi^i)_{1\leq i\leq n}$, where π^i denotes the proportion of wealth invested in the asset $S^i, i = 1, \cdots, n$. The wealth process associated to an initial capital $x > 0$ and the trading strategy π is defined by the equation

$$X_t^\pi \quad := \quad x + \int_0^t \pi_u X_{u-}^\pi dR_u, \ t \in [0,T].$$

Given the initial capital $x \in (0, +\infty)$, a trading strategy π is said to be admissible if and only if $X_t^\pi \geq 0, t \in [0,T]$. We denote by $\mathcal{A}(x)$ the set of admissible trading strategies for the initial capital x.

We consider an investor in the market endowed with an initial capital $x > 0$. Let $U : (0, +\infty) \to \mathbb{R}$ be a strictly increasing, strictly concave and continuously differentiable function measuring the utility of the investor. We extend U by setting $U(z) = -\infty$ if $z \leq 0$. Our investor aims to maximize his expected utility from terminal wealth. This results in solving an optimal stochastic control problem with value function u given by

$$u(x) \quad = \quad \sup_{\pi \in \mathcal{A}(x)} \mathbb{E}\left[U(X_T^\pi)\right], \tag{2.4}$$

with the convention $\mathbb{E}\left[U(X_T^\pi)\right] = -\infty$ if $\mathbb{E}\left[(U(X_T^\pi))^-\right] = +\infty$. The goal of the investor is to look for $\nu \in \mathcal{A}(x)$ such that $u(x) = \mathbb{E}\left[U(X_T^\nu)\right]$. Such a trading strategy will be referred to as an optimal trading strategy and X^ν as the optimal wealth process.

The main tool to show the existence of an optimal wealth process is provided by the convex or martingale duality methods of [KS99, KLSX91]. This approach exploits the duality relation between the set of attainable claims and the set of martingale measures to identify a suitable dual problem to (2.4). The domain $\mathcal{Y}$ of the dual problem is given by the set of nonnegative semimartingales Y such that $X^\pi Y$ is a supermartingale for every $\pi \in \mathcal{A}(1)$, i.e.

$$\mathcal{Y} \quad := \quad \{Y \geq 0 \mid Y_0 = 1 \text{ and } X^\pi Y \text{ is a supermartingale for every } \pi \in \mathcal{A}(1)\}.$$

Note that for every $\mathbb{Q} \in \mathcal{M}^e(S), Y = \left(\mathbb{E}\left[d\mathbb{Q}/d\mathbb{P} | \mathcal{F}_t\right]\right)_{t \in [0,T]} \in \mathcal{Y}$. Hence $\mathcal{Y} \neq \emptyset$. Moreover, every $Y \in \mathcal{Y}$ is a supermartingale since $\pi = (0, \cdots, 0) \in \mathcal{A}(1)$. We denote by $\mathcal{Y}^*$ the following subset of $\mathcal{Y}$:

$$\mathcal{Y}^* \quad = \quad \{Y \in \mathcal{Y} : Y > 0\}.$$

For $y > 0$, we define $\mathcal{Y}(y) = y\mathcal{Y}$ and $\mathcal{Y}^*(y) = y\mathcal{Y}^*(y)$. The value function v of the dual problem to (2.4) is given by

$$v(y) \quad = \quad \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[V\left(Y_T\right)\right], y > 0, \tag{2.5}$$

where for $y > 0$, $V(y) = \sup_{x>0}\left(U(x) - xy\right)$. We refer to $Y(y)$ attaining the inf in (2.5) as the dual optimizer. To ensure that an optimal wealth process for (2.4) and the dual optimizer for (2.5) exist, we work under the following standing assumption:

Assumption 2.2.11. *We suppose that :*

i) $\lim_{x \to 0} U^{'}(x) = +\infty$ *and* $\lim_{x \to +\infty} U^{'}(x) = 0$.

ii) $u(y) < +\infty$ *for some* $y > 0$.

iii) U *has asymptotic elasticity strictly less than one, i.e.* $AE[U] := \limsup\limits_{x \to +\infty} \frac{xU^{'}(x)}{U(x)} < 1$.

We now give the main statement in [KS99] that will be needed in the sequel.

Theorem 2.2.12. *[KS99, Theorem 2.2.] The following assertions are valid:*

1. u *and* v *are finite and continuously differentiable on* $(0, +\infty)$.
2. *For every* $(x, y) \in (0, +\infty) \times (0, +\infty)$*, there exists a unique pair* $(\nu, Y(y)) \in \mathcal{A}(x) \times \mathcal{Y}(y)$ *such that*

$$u(x) = \mathbb{E}\left[U(X_T^\nu)\right] \text{ and } v(y) = \mathbb{E}\left[V(Y_T(y))\right].$$

Moreover, if x *and* y *satisfy the relation* $y = u^{'}(x)$*, then*

$$Y_T(y) \quad = \quad U^{'}(X_T^\nu), \tag{2.6}$$

and $X^\nu Y(y)$ *is a uniformly integrable martingale.*

Remark 2.2.13. *Let $x > 0$ and $y = u^{'}(x)$. Note that by (2.6) and the properties of $U^{'}$, $X_T^{\nu} Y_T(y)$ is strictly positive. We deduce from the minimum principle for nonnegative supermartingales that $X^{\nu} Y(y)$ and $X_{-}^{\nu} Y_{-}(y)$ are strictly positive.*

2.3 The generalized opportunity process

We introduce and study the *generalized opportunity process*. First we consider the more familiar case of power utilities in Section 2.3.1 and recall the main results. The case of general utilities is introduced in Section 2.3.2 and its properties are analyzed in Section 2.4.

2.3.1 The opportunity process for power utility maximization

Let $p \in (-\infty, 0) \cup (0, 1)$ and $q = \frac{p}{p-1}$. Our goal in this section is to recall the notion of the *opportunity process* and its properties in the context of power utilities, i.e.

$$U(z) \quad = \quad \frac{z^p}{p}, \; z > 0.$$

The convex conjugate V of U is given by $V(z) = \frac{1-p}{p} z^{\frac{p}{p-1}} = -\frac{1}{q} z^q, z > 0$. We will denote by $\overline{u}_p$ the corresponding value function and by $\overline{v}_p$ the dual value function. We fix $x > 0$ and $y = \overline{u}_p^{'}(x)$.

Remark 2.3.1. *Note that $\overline{u}_p < +\infty$ is always satisfied for $p < 0$. For $p \in (0, 1)$, a sufficient condition for the finiteness of $\overline{u}_p$ is to have some element $Y \in \mathcal{Y}^*$ satisfy $\mathbb{E}\left[(Y_T)^{\frac{p}{p-1}}\right] < +\infty$ or $(A_{\frac{1}{p}})$. Indeed in this case we have for every $y > 0$*

$$\overline{v}_p(y) \quad \leq \quad \mathbb{E}\left[V(yY_T)\right] = \frac{1-p}{p} y^{\frac{p}{p-1}} \mathbb{E}\left[(Y_T)^{\frac{p}{p-1}}\right] = \frac{1-p}{p} y^{\frac{p}{p-1}} \mathbb{E}\left[\left(\frac{1}{Y_T}\right)^{\frac{1}{\frac{1}{p}-1}}\right] < +\infty.$$

We deduce from [KS03, Theorem 2] that $\overline{u}_p < +\infty$.

Due to the homogeneity of U, the direct stochastic control approach of the dynamic programming principle [EKQ95] is more feasible to address the primal problem (2.4).
Let $\pi \in \mathcal{A}(x), t \in [0, T]$. We set $\mathcal{A}_t(x, \pi) = \{\theta \in \mathcal{A}(x) \quad | \quad \theta_s = \pi_s, s \in [0, t]\}$ and

$$\overline{u}_p(t, X_t^{\pi}) \quad = \quad \operatorname*{ess\,sup}_{\theta \in \mathcal{A}_t(x,\pi)} \mathbb{E}\left[\frac{1}{p}(X_T^{\theta})^p \middle| \mathcal{F}_t\right], \; t \in [0, T].$$

$\overline{u}_p(t, X_t^{\pi})$ represents the maximal value the investor will obtain if he starts trading at time t with capital X_t^{π}. It follows from the dynamic programming principle that the family of random variables $\{\overline{u}_p(t, X_t^{\pi}), t \in [0, T]\}$ is a supermartingale (i.e. $\mathbb{E}\left[\overline{u}_p(t, X_t^{\pi}) \middle| \mathcal{F}_s\right] \leq \overline{u}_p(s, X_s^{\pi}), s \leq t \leq T$) and can be aggregated by a unique càdlàg supermartingale $\overline{u}_p(\cdot, X^{\pi})$. The following proposition from [Nut10] gives the precise description of $\overline{u}_p(\cdot, X^{\pi})$ and the characterization of the optimal trading strategy ν.

Proposition 2.3.2. *[Nut10, Propositions 3.1, 3.4 and 4.2] There exists a unique càdlàg semimartingale $L(p)$ such that for any $\pi \in \mathcal{A}(x)$*

$$\overline{u}_p(t, X_t^{\pi}) \quad = \quad \frac{1}{p}(X_t^{\pi})^p L_t(p), \; t \in [0, T].$$

In addition the following properties are satisfied:

i) For every $\pi \in \mathcal{A}(x)$, $\frac{1}{p}(X^{\pi})^p L(p)$ is a supermartingale and $\frac{1}{p}(X^{\nu})^p L(p)$ is a martingale.

ii) $L(p)$ *is a supermartingale if* $p \in (0,1)$ *and submartingale if* $p \in (-\infty, 0)$.

iii) $L(p), L_{-}(p) > 0$.

iv) The dual optimizer is given by $Y(y) = (X^{\nu})^{p-1} L(p)$.

The process $L(p)$ defined in Proposition 2.3.2 is the *opportunity process* for power utility maximization. Taking $\pi = 0$ and $x = 1$ in Proposition 2.3.2, we have $L_t(p) = p\overline{u}_p(t,1)$, $t \in [0,T]$. Thus $L_t(p)$ describes the maximal amount the investor would obtain if he began trading with initial capital 1 at time $t \in [0,T]$. This justifies the term *opportunity process*. It is independent of the initial capital. Proposition 2.3.2 shows that the knowledge of $L(p)$ is sufficient and necessary to describe all relevant objects for power utility maximization: optimal trading strategy, value function and dual optimizer.

Remark 2.3.3. $L(p)$ *has terminal value* 1*. Thus by Proposition 2.3.2 ii),* $L(p) \geq 1$ *for* $p \in (0,1)$ *and* $L(p) \leq 1$ *for* $p \in (-\infty, 0)$.

A key property of the opportunity process is that up to a power transformation, it has representation in terms of the dual optimizer and it is also the the value process to a control problem with control variables given by supermartingale deflators. The following result from [Nut10] gives the precise representation.

Proposition 2.3.4. *Let* $p \in (-\infty, 0) \cup (0,1)$ *and* $U(z) = \frac{z^p}{p}, z > 0$*. Let* $L(p)$ *be opportunity process defined in Proposition 2.3.2 and* $\widehat{Y}(y)$ *the dual optimizer. Then for every* $t \in [0,T]$ *we have*

$$(L_t(p))^{\frac{1}{1-p}} = \mathbb{E}\left[\left(\frac{Y_T(y)}{Y_t(y)}\right)^{\frac{p}{p-1}}\middle|\mathcal{F}_t\right] \quad = \quad \begin{cases} \operatorname{ess\,inf}_{Y \in \mathcal{Y}^*} \mathbb{E}\left[\left(\frac{Y_T}{Y_t}\right)^{\frac{p}{p-1}}\middle|\mathcal{F}_t\right], p \in (0,1), \\ \operatorname{ess\,sup}_{Y \in \mathcal{Y}^*} \mathbb{E}\left[\left(\frac{Y_T}{Y_t}\right)^{\frac{p}{p-1}}\middle|\mathcal{F}_t\right], \ p \in (-\infty, 0). \end{cases}$$

Proof. See Propositions 4.3 and 4.4. in [Nut10]. □

Let $p \in (-\infty, 0) \cup (0,1)$. An important feature of the representation given by Proposition 2.3.4 is the fact that one can infer the integrability properties of $L(p)$ such as the moments of its running maximum or the boundedness away from 0 and ∞, by analyzing a corresponding property for an arbitrary element in the space of supermartingale deflators. The following result from [Nut10] illustrates this fact for the boundedness of $L(p)$ away from 0 and ∞, which corresponds to the condition (b_q^-) for some $q \in (-\infty, 0) \cup (0,1)$ stated for some element in $\mathcal{Y}^*$.

Proposition 2.3.5. *[Nut10, Proposition 4.5] Let* $p \in (0,1)$*. Then the following assertions are equivalent:*

i) $L(p)$ *is bounded away from zero and infinity.*

ii) $Y(y)$ *satisfies* $(A_{\frac{1}{p}})$.

iii) $(A_{\frac{1}{p}})$ *is satisfied for some* $Y \in \mathcal{Y}^*$.

For $p < 0$*, the above assertions remain equivalent with* $(A_{\frac{1}{p}})$ *replaced by* (b_q^-) *where* $q = \frac{p}{p-1} \in (0,1)$.

2.3.2 The generalized opportunity process for utility maximization

From now on, we fix $x > 0$ and $y = u^{'}(x)$. Denote $\widehat{X} := X^{\nu}$ and $\widehat{Y} := Y(y)$ and let $p \in (-\infty, 0) \cup (0, 1)$. We have observed in the previous section that the essential feature of homogeneity for power utility functions reduces the study of the primal and dual problems (2.4-2.5) to that of the so-called *opportunity process* $L(p)$ which describes $\nu, \widehat{X}$ and $\widehat{Y}$. A typical utility function U does not possess the homogeneity of the power function. Thus the decoupling feature of the dynamic value process is not satisfied. In an attempt to describe $\nu, \widehat{X}$ and $\widehat{Y}$ and to analyze their properties, we have to rely on a different process. For $U(z) = \frac{z^p}{p}, z > 0$, a distinguished property of the opportunity process $L(p)$ is the multiplicative structure[1] it confers to $\widehat{Y}$, i.e. $\widehat{Y} = \widehat{X}^{p-1}L(p) = U^{'}(\widehat{X})L(p)$. Such a multiplicative structure of $\widehat{Y}$ is valid for more general utility functions U. This is a consequence of the martingale property of the product $\widehat{X}\widehat{Y}$ and the equality $\widehat{Y}_T = U^{'}(\widehat{X}_T)$. Indeed for every $\sigma \in \mathcal{T}(\mathbb{F})$, we have

$$\begin{aligned}\widehat{Y}_{\sigma} &= \frac{\widehat{X}_{\sigma}\widehat{Y}_{\sigma}}{\widehat{X}_{\sigma}} = \frac{\mathbb{E}\left[\widehat{X}_T\widehat{Y}_T|\mathcal{F}_{\sigma}\right]}{\widehat{X}_{\sigma}} = \frac{\mathbb{E}\left[\widehat{X}_T U^{'}(\widehat{X}_T)|\mathcal{F}_{\sigma}\right]}{\widehat{X}_{\sigma}} \\ &= U^{'}(\widehat{X}_{\sigma})\frac{\mathbb{E}\left[\widehat{X}_T U^{'}(\widehat{X}_T)|\mathcal{F}_{\sigma}\right]}{\widehat{X}_{\sigma}U^{'}(\widehat{X}_{\sigma})}.\end{aligned}$$

The equality above shows that there is a strong coupling between $\widehat{X}$ and $\widehat{Y}$ encoded by L, where

$$L := \frac{\widehat{Y}}{U^{'}(\widehat{X})}. \tag{2.7}$$

The process L depends only on $\widehat{X}$ and provides a representation of $\widehat{Y}$ via $\widehat{Y} = U^{'}(\widehat{X})L$. Following Proposition 2.3.2 we have $L = L(p)$ for U of power type with relative risk aversion $1 - p$. For this reason, we will refer to L as the *generalized opportunity process* for utility maximization. Note that L is not a reduced form of dynamic value function except for power utilities and the terminology *generalized opportunity process* is for consistency purposes. As already pointed in the introduction, the process L has appears already in [HHI$^+$14, ST14] in the context of utility maximization with random endowment as the backward component of a system of forward backward stochastic differential equations whose solution provide a tractable expression of the optimal trading strategy. The authors assume the asset price processes to be continuous and U to be three times continuously differentiable. Moreover, they only provide integrability properties of the optimizers (optimal wealth process $\widehat{X}$, dual optimizer $\widehat{Y}$ and optimal trading strategy ν) in the special case of a complete market. Our goal in this chapter is to present the integrability properties of L in the framework of a general semimartingale model S for the asset price processes and less restrictions on U, and to link these properties to those of the optimizers.

In order to study the integrability of L, we will rely on the following growth condition of U introduced in [KW16] to derive a sufficient condition for the dual optimizer $\widehat{Y}$ to satisfy the Muckenhoupt condition (A_r) for some $r > 1$.

Definition 2.3.6. *Let $a, b, C \in (0, +\infty)$ with $a \le b$ and $C \ge 1$. We say that U satisfies the inequality denoted by $(G_{a,b,C})$ if for every $x, y \in (0, +\infty)$ with $x \le y$ we have*

$$\frac{1}{C}\left(\frac{y}{x}\right)^a \le \frac{U^{'}(x)}{U^{'}(y)} \le C\left(\frac{y}{x}\right)^b. \tag{2.8}$$

[1] This decomposition of the dual optimizer was actually used in [KMK10] to define the opportunity process $L(p)$.

Note that U satisfies the growth condition $(G_{a,b,C})$ as soon as its coefficient of relative risk aversion has lower bound a and upper bound b [2], i.e.

$$a \le -\frac{tU^{''}(t)}{U'(t)} \le b < \infty, \ t > 0.$$

Example 2.3.7. *The condition $(G_{a,b,C})$ holds for the following utility functions:*

i) $U(z) = \log z, \ z > 0$. *We have* $a = b = 1$.

ii) $U(z) = \frac{z^p}{p}, \ z > 0, \ p \in (-\infty, 0) \cup (0, 1)$. $(G_{a,b,C})$ *holds with* $a = b = 1 - p$.

iii) $U(z) = \frac{z^p}{p} + \frac{z^q}{q}, \ z > 0, \ p, q \in (-\infty, 0) \cup (0, 1)$ *with* $p \le q$. *Then* $a = 1 - q, b = 1 - p$.

It easy to show that $(G_{a,b,C}) \Rightarrow (G_{a^{'},b,C}) \Rightarrow (G_{a^{'},b^{'},C})$ for $a, a^{'}, b, b^{'} \in (0, \infty)$ with $0 < a^{'} \le a \le b \le b^{'}$. Thus if $(G_{a,b,C})$ holds for some $a, b \in (0, +\infty)$, we can assume w.l.o.g. that $a \in (0, 1)$. However, the case $a \ge 1$ is interesting to study in its own rights as it requires weaker conditions for example for $\widehat{Y}$ to be of class (D) and for L to be bounded (see Theorem 2.5.1 and Proposition 2.5.6).

We close this section by observing that under the condition $(G_{a,b,C})$, the finiteness of the value function u reduces to that of power utilities.

Lemma 2.3.8. *Suppose that U satisfies $(G_{a,b,C})$ with $a \le b$. The value function u is finitely valued if*

i) $a \neq 1$ *and* $\overline{u}_{1-a}$ *is finitely valued,*

ii) $a = 1$ *and there exists* $p \in (0, 1)$ *such that* $\overline{u}_p$ *is finitely valued.*

Proof. Let $p \in (0, 1)$. Set $\overline{x} = (\frac{1}{p})^{\frac{1}{p}}$ and $x_0 = 1 + \overline{x}$. The condition $(G_{a,b,C})$ implies that for $x \ge x_0$

$$U^{'}(x) \le CU^{'}(x_0)x_0^a \frac{1}{x^a}. \tag{2.9}$$

Now assume that $a \neq 1$ and $\overline{u}_{1-a} < +\infty$. Integrating (2.9) and using the monotonicity property of U yields the following upper bound for U

$$U(x) \le U(x_0) + CU^{'}(x_0)x_0^a \left(\frac{x^{1-a}}{1-a} - \frac{x_0^{1-a}}{1-a} \right) 1_{\{x \ge x_0\}}, \ x > 0. \tag{2.10}$$

We consider the case $a > 1$. Then $1 - a < 0$, and (2.10) gives for $x > 0$,

$$U(x) \le U(x_0) + CU^{'}(x_0)\frac{x_0}{a-1}. \tag{2.11}$$

Since U is bounded from above, u is finitely valued.
We consider the case $a < 1$. Then $1 - a > 0$, and (2.10) implies that

$$U(x) \le U(x_0) + CU^{'}(x_0)x_0^a \frac{x^{1-a}}{1-a}, \ x > 0. \tag{2.12}$$

[2] Indeed $a \le -\frac{U^{''}(t)t}{U^{'}(t)} \le b, t > 0$ is equivalent to $\frac{a}{t} \le -\frac{U^{''}(t)}{U^{'}(t)} \le \frac{b}{t}, t > 0$. Thus integrating the latter inequalities between x and y, with $x \le y$ we obtain U satisfies $(G_{a,b,C})$ with $C = 1$.

We infer from (2.12) that for $y > 0$, we have $u(y) \leq U(x_0) + CU^{'}(x_0)x_0^a\overline{u}_{1-a}(y) < +\infty$.
Now assume that $a = 1$ and $\overline{u}_p < +\infty$. Integrating once more (2.9) and using the monotonicity property of U, we obtain for $x > 0$

$$U(x) \leq U(x_0) + CU^{'}(x_0)x_0\big(\log x - \log x_0\big)1_{\{x \geq x_0\}}.$$

Note that $\log x_0 \geq 0$, and due to the choice of $\overline{x}$, we have $\log x \leq x^p$ for $x \geq x_0$. It follows from the previous inequality that for $x > 0$

$$U(x) \quad \leq \quad U(x_0) + CU^{'}(x_0)x_0x^p. \tag{2.13}$$

Hence for every $y > 0$, we have $u(y) \leq U(x_0) + CU^{'}(x_0)x_0p\overline{u}_p(y) < +\infty$. □

2.4 A priori estimates of the generalized opportunity process

In this section, we use the condition $(G_{a,b,C})$ to derive estimates of the process L in terms of the opportunity processes $L(p), p \in (-\infty, 0) \cup (0, 1)$. As a result, our analysis of the integrability properties of L such as moments or uniform bounds will be reduced to those of $L(p)$ or more precisely to that of some element in the set of supermartingale deflators $\mathcal{Y}^*$. The *a priori* estimates of L will prove very useful in Chapter 3 to provide a normed space of the solutions to the non-standard system of fully coupled FBSDEs describing the joint dynamics of $(\widehat{X}, L)$.

We begin our study by stating a useful lemma from functional calculus which we will often rely on to obtain estimates of L.

Lemma 2.4.1. *Let $C_1, C_2, C_3 \in [0, \infty[$ and $\alpha, \beta \in [0, 1)$ with $C_1\alpha + C_2\beta > 0$. Let $f : (0, \infty) \ni z \mapsto z - C_1z^\alpha - C_2z^\beta - C_3$. Then there exits a unique $\bar{z} > 0$ such $f(\bar{z}) = 0$. Moreover $f(z) \leq 0$ if and only if $z \leq \bar{z}$.*

Proof. Clearly f is continuously differentiable on $(0, \infty)$ with $f^{'}(z) = 1 - C_1\alpha z^{\alpha-1} - C_2\beta z^{\beta-1}, z > 0$. Since $\lim_{z\to 0^+} f(z) \leq 0$ and $\lim_{z\to+\infty} f(z) = +\infty$, f changes sign on $]0, \infty[$. Therefore there exists $\bar{z} > 0$ such that $f(\bar{z}) = 0$. Observe that $f^{'}$ is a strictly increasing function, $\lim_{z\to 0^+} f^{'}(z) = -\infty$ and $\lim_{z\to+\infty} f^{'}(z) = 1$. Thus there exits $z_1 > 0$ such that $f^{'}(z_1) = 0$. We deduce that f is strictly decreasing (resp. increasing) on $]0, z_1]$ (resp. $[z_1, \infty[$). Consequently, $f(z_1) \leq \lim_{z\to 0^+} f(z) \leq 0$. As f is negative on $]0, z_1]$, $z_1 < \bar{z}$. Now f is strictly increasing on $[z_1, \infty[$. We deduce that $\bar{z}$ is unique and $f(z) \leq 0$ if and only if $z \leq \bar{z}$. □

The following proposition shows that for $a \geq 1$, the process L is "almost" a submartigale while for $b < 1$, it is almost a "supermartingale".

Proposition 2.4.2. *Suppose that U satisfies $(G_{a,b,C})$ with $0 < a \leq b$. The following assertions hold:*

i) In the case $1 \leq a$, there exists a constant C_1 depending only on b and C such that

$$L_s \leq \max\{C_1, 1\}\mathbb{E}\left[L_t|\mathcal{F}_s\right] \;\; s, t \in [0, T], \;\; s \leq t.$$

In particular, $L \leq \max\{C_1, 1\}$.

ii) In the case $b < 1$, there exists a constant C_2 depending only on C and a such that

$$\mathbb{E}\left[L_t|\mathcal{F}_s\right] \quad \leq \quad C_2L_s, \;\; s, t \in [0, T], \;\; s \leq t.$$

Moreover, $C_2L \geq 1$.

Proof. Note that $L_T = 1$, therefore the bounds $L \leq \max\{C_1, 1\}$ and $C_2 L \geq 1$ are consequences of the conditional inequalities on which we focus. Let $s, t \in [0, T],\ s \leq t$.

i) Assume that $a \geq 1$. Then $1 \leq a \leq b$. Using the tower property and the definition of L, we have

$$\begin{aligned} L_s &= \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_t U'(\widehat{X}_t)} \frac{\widehat{X}_t U'(\widehat{X}_t)}{\widehat{X}_s U'(\widehat{X}_s)} \middle| \mathcal{F}_s\right] = \mathbb{E}\left[\mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_t U'(\widehat{X}_t)} \middle| \mathcal{F}_t\right] \frac{\widehat{X}_t U'(\widehat{X}_t)}{\widehat{X}_s U'(\widehat{X}_s)} \middle| \mathcal{F}_s\right] \\ &= \mathbb{E}\left[L_t \frac{\widehat{X}_t U'(\widehat{X}_t)}{\widehat{X}_s U'(\widehat{X}_s)} \middle| \mathcal{F}_s\right]. \end{aligned}$$

Due to $(G_{a,b,C})$ we have

$$\begin{aligned} \frac{\widehat{X}_t U'(\widehat{X}_t)}{\widehat{X}_s U'(\widehat{X}_s)} &= \frac{\widehat{X}_t U'(\widehat{X}_t)}{\widehat{X}_s U'(\widehat{X}_s)} 1_{\{\widehat{X}_t \leq \widehat{X}_s\}} + \frac{\widehat{X}_t U'(\widehat{X}_t)}{\widehat{X}_s U'(\widehat{X}_s)} 1_{\{\widehat{X}_t > \widehat{X}_s\}} \\ &\leq C^{\frac{1}{b}} \left(\frac{U'(\widehat{X}_t)}{U'(\widehat{X}_s)}\right)^{1-\frac{1}{b}} 1_{\{\widehat{X}_t \leq \widehat{X}_s\}} + C \left(\frac{\widehat{X}_s}{\widehat{X}_t}\right)^{a-1} 1_{\{\widehat{X}_t > \widehat{X}_s\}}. \end{aligned}$$

The above estimate and the representation of L_s leads to the upper bound:

$$L_s \leq C^{\frac{1}{b}} \mathbb{E}\left[L_t \left(\frac{U'(\widehat{X}_t)}{U'(\widehat{X}_s)}\right)^{1-\frac{1}{b}} 1_{\{\widehat{X}_t \leq \widehat{X}_s\}} \middle| \mathcal{F}_s\right] + C\mathbb{E}\left[L_t \left(\frac{\widehat{X}_s}{\widehat{X}_t}\right)^{a-1} 1_{\{\widehat{X}_t > \widehat{X}_s\}} \middle| \mathcal{F}_s\right]. \tag{2.14}$$

We consider the following two cases:

Case 1: $b = 1$. Then $a = b = 1$. As $L_T = 1$, taking $t = T$, we infer from (2.14) that

$$L_s \leq \max\{C, C^{\frac{1}{b}}\}.$$

For $t \in [s, T)$, (2.14) gives $L_s \leq \max\{C, C^{\frac{1}{b}}\} \mathbb{E}\left[L_t | \mathcal{F}_s\right]$. We obtain i) with $C_1 = \max\{C, C^{\frac{1}{b}}\}$.

Case 2: $b > 1$. As $\frac{1}{b} + \frac{b-1}{b} = 1$, Hölder's inequality yields

$$\begin{aligned} \mathbb{E}\left[L_t \left(\frac{U'(\widehat{X}_t)}{U'(\widehat{X}_s)}\right)^{1-\frac{1}{b}} 1_{\{\widehat{X}_t \leq \widehat{X}_s\}} \middle| \mathcal{F}_s\right] &\leq \mathbb{E}\left[L_t^{\frac{1}{b}} \left(\frac{U'(\widehat{X}_t) L_t}{U'(\widehat{X}_s)}\right)^{1-\frac{1}{b}} \middle| \mathcal{F}_s\right] \\ &\leq \left(\mathbb{E}\left[L_t | \mathcal{F}_s\right]\right)^{\frac{1}{b}} \left(\mathbb{E}\left[\frac{U'(\widehat{X}_t) L_t}{U'(\widehat{X}_s)} \middle| \mathcal{F}_s\right]\right)^{\frac{b-1}{b}}. \end{aligned}$$

$U'(\widehat{X})L$ being a supermartingale, $\left(\mathbb{E}\left[\frac{U'(\widehat{X}_t) L_t}{U'(\widehat{X}_s)} \middle| \mathcal{F}_s\right]\right)^{\frac{b-1}{b}} \leq L_s^{\frac{b-1}{b}}$ and we deduce that

$$\mathbb{E}\left[L_t \left(\frac{U'(\widehat{X}_t)}{U'(\widehat{X}_s)}\right)^{1-\frac{1}{b}} 1_{\{\widehat{X}_t \leq \widehat{X}_s\}} \middle| \mathcal{F}_s\right] \leq \left(\mathbb{E}\left[L_t | \mathcal{F}_s\right]\right)^{\frac{1}{b}} L_s^{1-\frac{1}{b}}.$$

Since $a \geq 1$, $\left(\frac{\widehat{X}_s}{\widehat{X}_t}\right)^{a-1} 1_{\{\widehat{X}_s \leq \widehat{X}_t\}} \leq 1$. Thus $\mathbb{E}\left[L_t \left(\frac{\widehat{X}_s}{\widehat{X}_t}\right)^{a-1} 1_{\{\widehat{X}_t > \widehat{X}_s\}} \middle| \mathcal{F}_s\right] \leq \mathbb{E}\left[L_t | \mathcal{F}_s\right]$. We infer from (2.14) and the previous estimates that

$$L_s \leq C\mathbb{E}\left[L_t | \mathcal{F}_s\right] + C^{\frac{1}{b}} L_s^{1-\frac{1}{b}} \left(\mathbb{E}\left[L_t | \mathcal{F}_s\right]\right)^{\frac{1}{b}}. \tag{2.15}$$

We set $J_{s,t} = L_s / \mathbb{E}\left[L_t | \mathcal{F}_s\right]$. Inserting $J_{s,t}$ into (2.15), we obtain $J_{s,t} \leq C + C^{\frac{1}{b}} J_{s,t}^{1-\frac{1}{b}}$. By Lemma 2.4.1, $J_{s,t} \leq C_1$, where C_1 is the root of the equation

$$z = C + C^{\frac{1}{b}} z^{1-\frac{1}{b}},\ z > 0.$$

Since $J_{s,s} = 1$, we deduce that $L_s \leq \max\{C_1, 1\}\mathbb{E}[L_t|\mathcal{F}_s]$.

ii) Assume that $b < 1$. Then $0 < a \leq b$. We start by noting that

$$\mathbb{E}[L_t|\mathcal{F}_s] = \mathbb{E}\left[L_t 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}|\mathcal{F}_s\right] + \mathbb{E}\left[L_t 1_{\{\widehat{X}_s > \widehat{X}_t\}}|\mathcal{F}_s\right].$$

To prove the assertion, we provide an upper bound for $\mathbb{E}\left[L_t 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}|\mathcal{F}_s\right]$ and $\mathbb{E}\left[L_t 1_{\{\widehat{X}_s > \widehat{X}_t\}}|\mathcal{F}_s\right]$ separately.

We begin with $\mathbb{E}\left[L_t 1_{\{\widehat{X}_s > \widehat{X}_t\}}|\mathcal{F}_s\right]$. As U' is decreasing, $1_{\{\widehat{X}_s > \widehat{X}_t\}} \leq \left(\frac{U'(\widehat{X}_t)}{U'(\widehat{X}_s)}\right)^a 1_{\{\widehat{X}_s > \widehat{X}_t\}}$ and thus

$$\mathbb{E}\left[L_t 1_{\{\widehat{X}_s > \widehat{X}_t\}}|\mathcal{F}_s\right] \leq \mathbb{E}\left[L_t \left(\frac{U'(\widehat{X}_t)}{U'(\widehat{X}_s)}\right)^a 1_{\{\widehat{X}_s > \widehat{X}_t\}}\middle|\mathcal{F}_s\right] = \mathbb{E}\left[L_t^{1-a} \left(\frac{U'(\widehat{X}_t)L_t}{U'(\widehat{X}_s)}\right)^a 1_{\{\widehat{X}_s > \widehat{X}_t\}}\middle|\mathcal{F}_s\right].$$

Applying Hölder's inequality and the supermartingale property of $U'(\widehat{X})L$, we obtain

$$\mathbb{E}\left[L_t^{1-a} \left(\frac{U'(\widehat{X}_t)L_t}{U'(\widehat{X}_s)}\right)^a 1_{\{\widehat{X}_s > \widehat{X}_t\}}\middle|\mathcal{F}_s\right] \leq \mathbb{E}\left[L_t^{\frac{1}{p}} \left(\frac{U'(\widehat{X}_t)L_t}{U'(\widehat{X}_s)}\right)^{\frac{1}{q}}\middle|\mathcal{F}_s\right] = \left(\mathbb{E}[L_t|\mathcal{F}_s]\right)^{\frac{1}{p}} L_s^{\frac{1}{q}},$$

where $p = \frac{1}{1-a}, q = \frac{1}{a}$. We infer that

$$\mathbb{E}\left[L_t 1_{\{\widehat{X}_s > \widehat{X}_t\}}|\mathcal{F}_s\right] \leq \left(\mathbb{E}[L_t|\mathcal{F}_s]\right)^{1-a} L_s^a.$$

Next we provide an upper bound for $\mathbb{E}\left[L_t 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}|\mathcal{F}_s\right]$. Using the tower property, we have

$$\mathbb{E}\left[L_t 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}|\mathcal{F}_s\right] = \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_t U'(\widehat{X}_t)} 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}\middle|\mathcal{F}_s\right] = \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_s U'(\widehat{X}_s)} \frac{\widehat{X}_s U'(\widehat{X}_s)}{\widehat{X}_t U'(\widehat{X}_t)} 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}\middle|\mathcal{F}_s\right].$$

As U satisfies $(G_{a,b,C})$ with $b < 1$, $\frac{\widehat{X}_s U'(\widehat{X}_s)}{\widehat{X}_t U'(\widehat{X}_t)} 1_{\{\widehat{X}_s \leq \widehat{X}_t\}} \leq C\left(\frac{\widehat{X}_s}{\widehat{X}_t}\right)^{1-b} 1_{\{\widehat{X}_s \leq \widehat{X}_t\}} \leq C$. Therefore,

$$\mathbb{E}\left[L_t 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}|\mathcal{F}_s\right] = \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_s U'(\widehat{X}_s)} \frac{\widehat{X}_s U'(\widehat{X}_s)}{\widehat{X}_t U'(\widehat{X}_t)} 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}\middle|\mathcal{F}_s\right] \leq C\mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_s U'(\widehat{X}_s)}\middle|\mathcal{F}_s\right] = CL_s.$$

We recall that $\mathbb{E}[L_t|\mathcal{F}_s] = \mathbb{E}\left[L_t 1_{\{\widehat{X}_s \leq \widehat{X}_t\}}|\mathcal{F}_s\right] + \mathbb{E}\left[L_t 1_{\{\widehat{X}_s > \widehat{X}_t\}}|\mathcal{F}_s\right]$. Hence the above estimates yield

$$\mathbb{E}[L_t|\mathcal{F}_s] \leq CL_s + \left(\mathbb{E}[L_t|\mathcal{F}_s]\right)^{1-a} L_s^a. \tag{2.16}$$

Set $L_{s,t} = \mathbb{E}[L_t|\mathcal{F}_s]/L_s$. Then (2.16) entails that $L_{s,t} \leq C + L_{s,t}^{1-a}$. It follows from Lemma 2.4.1 that $L_{s,t} \leq C_2$ where C_2 is the root of the equation

$$z = C + z^{1-a}, z > 0.$$

As a result, $\mathbb{E}[L_t|\mathcal{F}_s] \leq C_2 L_s$. The proof is complete. □

Remark 2.4.3. *For S having continuous paths and for U three times continuously differentiable, it will be shown in Proposition 3.4.12 that L is indeed a submartingale for $a \geq 1$ and a supermartingale for $b < 1$. We will therefore have $C_1 = C_2 = 1$.*

The following lemma gives a pathwise inequality that leads to a lower bound for L provided it is bounded from above.

Lemma 2.4.4. *Suppose that U satisfies $(G_{a,b,C})$ with $0 < a \le b$. Let $\alpha \in (0,a)$ and β such that $0 < \beta < \frac{\alpha}{b}$. Then for every $\sigma \in \mathcal{T}(\mathbb{F})$, we have*

$$L_\sigma(-\alpha) \le \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha \middle| \mathcal{F}_\sigma\right] \quad \le \quad C^\beta L_\sigma^\beta + C^{\frac{\alpha}{a}} L_\sigma^{\frac{\alpha}{a}}. \tag{2.17}$$

Proof. By Proposition 2.3.2, $\widehat{X}^{-\alpha}L(-\alpha)$ is a submartingale with terminal value $\widehat{X}_T^{-\alpha}$. The submartingale martingale property gives $L_\sigma(-\alpha) \le \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha \middle| \mathcal{F}_\sigma\right]$ which proves the first half of the inequality (2.17). We now show the second half. First we show that

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \le C^\beta L_\sigma^\beta.$$

To this end, we choose $r > \frac{1}{1-\beta}$ and $\bar{r} = \frac{r}{r-1}$. An application of Hölder's inequality yields

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] &= \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha \left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^\beta \left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^\beta 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \\ &\le \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\alpha r} \left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\beta r} 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{r}} \left(\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^{\beta \bar{r}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{\bar{r}}}.\end{aligned}$$

Since $\beta\bar{r} < 1$ and $\widehat{Y}$ is a supermartingale, it follows from Jensen's inequality that

$$\left(\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^{\beta \bar{r}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{\bar{r}}} \quad \le \quad \left(\mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_\sigma} \middle| \mathcal{F}_\sigma\right]\right)^\beta \le 1.$$

Therefore

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \quad \le \quad \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\alpha r} \left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\beta r} 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{r}}. \tag{2.18}$$

Now $\widehat{Y} = U'(\widehat{X})L$. Due to $(G_{a,b,C})$, $\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\beta r} 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \le C^{\beta r} L_\sigma^{\beta r} \left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\beta b r} 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}}$. Inserting the latter into (2.18) and using the fact that $\alpha - \beta b > 0$, we obtain the following upper bound:

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \quad \le \quad C^\beta L_\sigma^\beta \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{r(\alpha-\beta b)} 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{1}{r}} \le C^\beta L_\sigma^\beta.$$

To obtain (2.17), it remains to show that $\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \le C^{\frac{\alpha}{a}} L_\sigma^{\frac{\alpha}{a}}$. We infer from $(G_{a,b,C})$ that

$$\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \quad = \quad C^{\frac{\alpha}{a}} \left(\frac{1}{C}\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^a\right)^{\frac{\alpha}{a}} 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \le C^{\frac{\alpha}{a}} \left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^{\frac{\alpha}{a}} 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}}.$$

But $\alpha < a$. Hence, using Jensen's inequality and the supermartingale property of $U'(\widehat{X})L$ we obtain

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \quad \le \quad C^{\frac{\alpha}{a}} \mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)L_\sigma}\right)^{\frac{\alpha}{a}} L_\sigma^{\frac{\alpha}{a}} 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \le C^{\frac{\alpha}{a}} L_\sigma^{\frac{\alpha}{a}}.$$

This proves (2.17). The proof is complete. □

We recall that for $a > 1$, L is bounded from above by Proposition 2.4.2 i). The following theorem shows that L is bounded away from 0 as soon as $L(p)$ is bounded away from 0 for some $p \in (-\infty, 0)$.

Theorem 2.4.5. *Suppose that U satisfies $(G_{a,b,C})$ with $a \geq 1$. Assume that there exists $q \in (0,1)$ and $Y \in \mathcal{Y}^*$ satisfying (b_q^-). Then there exist two strictly positive constants $\underline{l}, \overline{l}$ depending only on a, b and C such that:*

$$\underline{l} \leq L \leq \overline{l}. \tag{2.19}$$

Proof. Let $Y \in \mathcal{Y}^*$ satisfy (b_q^-). Then by Lemma 2.2.6 Y satisfies $\left(b_l^-\right)$ for all $l \in (0,1)$. It follows from Proposition 2.3.5 that $L(-\alpha)$ is bounded away from 0 for all $\alpha > 0$. Let $\alpha < a$ and $\beta \in (0, \frac{\alpha}{b})$.
We set $\kappa_\alpha = \inf_{t \in [0,T]} L_t(-\alpha) > 0$. We infer from Lemma 2.4.4 that for $t \in [0,T]$

$$\kappa_\alpha \leq L_t(-\alpha) \leq \mathbb{E}\left[\frac{X_t^\alpha}{X_T^\alpha}\middle|\mathcal{F}_t\right] \quad \leq \quad C^\beta L_t^\beta + C^{\frac{\alpha}{a}} L_t^{\frac{\alpha}{a}} \leq \max\{C^\beta, C^{\frac{\alpha}{a}}\} L_t^\beta \left(1 + L_t^{\frac{\alpha}{a}-\beta}\right). \tag{2.20}$$

Since $a \geq 1$, by Proposition 2.4.2 $L \leq \overline{l} = 1 + C_1$ where $C_1 > 0$ is a positive constant depending only on b and C. We deduce from (2.20) that $L \geq \underline{l}$ with $\underline{l}^\beta = \frac{\kappa_\alpha}{\max\{C^\beta, C^{\frac{\alpha}{a}}\}\left(1+(1+C_1)^{\frac{\alpha}{a}-\beta}\right)}$. □

Proposition 2.4.2 and Theorem 2.4.5 show that for $a \geq 1$, L and $L(1-a)$ possess the same properties in terms of uniform bounds. This will be further emphasized by Theorem 2.5.1 which gives a converse to Theorem 2.4.5 for the particular case where $a > 1$. For $a \in (0,1)$, we will only be able to show that L inherits the integrability properties of $L(1-a)$. The starting point of our analysis is the following lemma which establishes an inequality for L leading to a pathwise estimate for its upper bound.

Lemma 2.4.6. *Suppose that U satisfies $(G_{a,b,C})$ with $a \in (0,1)$ and $\overline{u}_{1-a} < +\infty$. Let $\gamma \in \mathbb{R}$ such that $0 < \gamma < \min\left\{\frac{1}{b}, 1\right\}$. Then for every $\sigma \in \mathcal{T}(\mathbb{F})$, we have*

$$L_\sigma \quad \leq \quad C^\gamma L_\sigma^{1-\gamma} + C L_\sigma(1-a). \tag{2.21}$$

Proof. Let $\gamma \in (0,1)$ satisfying $0 < \gamma < \min\left\{\frac{1}{b}, 1\right\}$. Let $\sigma \in \mathcal{T}(\mathbb{F})$. We recall that

$$L_\sigma \quad = \quad \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}}\middle|\mathcal{F}_\sigma\right] + \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}}\middle|\mathcal{F}_\sigma\right].$$

First we provide an upper bound of $\mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}}\middle|\mathcal{F}_\sigma\right]$. An application of $(G_{a,b,C})$ yields

$$\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \quad = \quad \frac{1}{\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \leq C\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^a 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \leq C\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{-a}.$$

We infer from the supermartingale property of $\widehat{X}^{1-a} L(1-a)$ that

$$\mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}}\middle|\mathcal{F}_\sigma\right] \quad \leq \quad C\mathbb{E}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{1-a}\middle|\mathcal{F}_\sigma\right] \leq C L_\sigma(1-a).$$

Next we give an upper bound of the term

$$\mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\frac{\widehat{X}_T}{\widehat{X}_\sigma}\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^\gamma \left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^{1-\gamma} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}}\right].$$

As U satisfies $(G_{a,b,C})$, we have $\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^\gamma 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \leq C^\gamma \left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\gamma b} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}}$. Consequently

$$\frac{\widehat{X}_T}{\widehat{X}_\sigma}\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^\gamma 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \quad \leq \quad C^\gamma \left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\gamma b - 1} = C^\gamma \left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{1-\gamma b} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \leq C^\gamma.$$

We deduce that

$$\mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] \quad \leq \quad C^\gamma \mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^{1-\gamma} \middle| \mathcal{F}_\sigma\right]. \tag{2.22}$$

Using Jensen's inequality and the supermartingale property of $\widehat{Y} = U'(\widehat{X})L$, we obtain

$$\mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^{1-\gamma} \middle| \mathcal{F}_\sigma\right] \quad \leq \quad \left(\mathbb{E}\left[\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)} \middle| \mathcal{F}_\sigma\right]\right)^{1-\gamma} = L_\sigma^{1-\gamma}\left(\mathbb{E}\left[\frac{U'(\widehat{X}_T)L_T}{U'(\widehat{X}_\sigma)L_\sigma} \middle| \mathcal{F}_\sigma\right]\right)^{1-\gamma} \leq L_\sigma^{1-\gamma}.$$

We infer from (2.22) that $\mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] \leq C^\gamma L_\sigma^{1-\gamma}$. We obtain (2.21) by summing up the two upper bounds. The proof is complete. □

The following proposition shows that for $a \in (0,1)$, L inherits the integrability properties of $L(1-a)$.

Proposition 2.4.7. *Suppose that U satisfies $(G_{a,b,C})$ with $a \in (0,1)$ and $\overline{u}_{1-a} < +\infty$. Then:*

A1. L is integrable, i.e. $\mathbb{E}[L_t] < +\infty, \forall\, t \in [0,T]$.

A2. L is bounded from above if $L(1-a)$ is bounded from above.

If additionally L is a semimartingale, then :

A3. L is a special semimartingale.

A4. If $L(1-a) \in \mathcal{S}^r$ for some $r > 0$, then $L \in \mathcal{S}^r$.

Proof. Let $\gamma \in \mathbb{R}$ such that $0 < \gamma < \min\{1, \frac{1}{b}\}$. For a real valued càdlàg process Γ, Γ^* is defined as follows $\Gamma^*_t = \sup_{s\in[0,t]} |\Gamma_s|,\ t \in [0,T]$.
A1. Let $t \in [0,T]$. By Lemma 2.4.6 we have $L_t \leq C^\gamma L_t^{1-\gamma} + CL_t(1-a)$. Integrating, and applying Jensen's inequality, we obtain

$$\mathbb{E}[L_t] \quad \leq \quad C^\gamma (\mathbb{E}[L_t])^{1-\gamma} + C\mathbb{E}[L_t(1-a)].$$

$L(1-a)$ being a supermatingale, we have $\mathbb{E}[L_t(1-a)] < +\infty$. We deduce from the inequality above that $\mathbb{E}[L_t]$ satisfies the inequality $z \leq C^\gamma z^{1-\gamma} + C\mathbb{E}[L_t(1-a)], z > 0$. Therefore by Lemma 2.4.1, $\mathbb{E}[L_t] < +\infty$. As t is arbitrary, L is integrable.
A2. Let $\epsilon = ||L(1-a)||_\infty < +\infty$. By Lemma 2.4.6, for each $t \in [0,T]$,

$$L_t \quad \leq \quad C^\gamma L_t^{1-\gamma} + C\epsilon.$$

We infer from Lemma 2.4.1 that $L_t \leq K_\epsilon$ where K_ϵ is the root of the equation

$$z = C^\gamma z^{1-\gamma} + C\epsilon, z > 0. \tag{2.23}$$

As K_ϵ is independent of t, $\omega \in \Omega$ and L has càdlàg paths[3], it follows that $\sup_{t\in[0,T]} L_t \leq K_\epsilon$. Thus L is bounded from above.

[3] L has càdlàg paths since $\widehat{X}$ and $\widehat{Y}$ have càdlàg paths and U' is continuous.

A3. Assume that L is a semimartingale. $L(1-a)$ is a special semimartingale by Proposition 2.3.2. Hence $L^*(1-a)$ is locally integrable, see [Pro04, Theorem 33, Chapter 3]. Let $(\sigma_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}(\mathbb{F})$ be a localizing sequence for $L^*(1-a)$. We will show that $(\sigma_n)_{n\in\mathbb{N}}$ is also a localizing sequence for L^*. Let $t \in [0,T]$ and $n \in \mathbb{N}$. By Lemma 2.4.6 we have

$$L^*_{t\wedge\sigma_n} \quad\leq\quad C^\gamma \left(L^*_{t\wedge\sigma_n}\right)^{1-\gamma} + CL^*_{t\wedge\sigma_n}(1-a).$$

Integrating the above inequality and applying Jensen's inequality we obtain

$$\mathbb{E}\left[L^*_{t\wedge\sigma_n}\right] \quad\leq\quad C^\gamma \left(\mathbb{E}\left[L^*_{t\wedge\sigma_n}\right]\right)^{1-\gamma} + C\mathbb{E}\left[L^*_{t\wedge\sigma_n}(1-a)\right].$$

Now $\mathbb{E}\left[L^*_{t\wedge\sigma_n}(1-a)\right] = \kappa < +\infty$. Clearly $\mathbb{E}\left[L^*_{t\wedge\sigma_n}\right]$ satisfies the inequality $z \leq C^\gamma z^{1-\gamma} + C\kappa$. Again by Lemma 2.4.1, $\mathbb{E}\left[L^*_{t\wedge\sigma_n}\right] < +\infty$ and $L^*_{t\wedge\sigma_n}$ is integrable. Consequently, $(\sigma_n)_{n\in\mathbb{N}}$ is a localizing sequence for L^* and thus L is a special semimartingale.
A4. Assume $\delta = \mathbb{E}\left[\left(L^*_T(1-a)\right)^r\right] < +\infty$. By Lemma 2.4.6, we have $L^*_T \leq \left(L^*_T\right)^{1-\gamma} + L^*_T(1-a)$. Using the binomial inequality $(x+y)^r \leq 2^r(x^r+y^r)$ for $x, y > 0$ and Jensen's inequality, we deduce from the preceding inequality that

$$\mathbb{E}\left[\left(L^*_T\right)^r\right] \leq 2^r \left(\mathbb{E}\left[\left(L^*_T\right)^r\right]\right)^{1-\gamma} + 2^r\delta. \tag{2.24}$$

As $\gamma \in (0,1)$, there exists a unique $z^* > 0$ solution to the equation

$$z = 2^r z^{1-\gamma} + 2^r\delta, \;\; z > 0.$$

We infer from (2.24) and Lemma 2.4.1 that $\mathbb{E}\left[\left(L^*_T\right)^r\right] \leq z^*$ which implies that $L \in \mathcal{S}^r$. □

Remark 2.4.8. *Note that L is semimartingale if for example U' is twice differentiable, convex or concave. If additionally, $1/U'(\widehat{X})$ is locally bounded, then L is a special semimartingale. The condition $(G_{a,b,C})$ is not needed, see for example Lemma 3.3.4.*

Applying Lemma 2.4.4 and Proposition 2.4.7, we obtain the following Theorem which shows that L is bounded away from 0 and ∞, if $L(1-a)$ possesses such a property.

Theorem 2.4.9. *Suppose that U satisfies $(G_{a,b,C})$ with $a \in (0,1)$ and there exists $Y \in \mathcal{Y}^*$ which satisfies $\left(A_{\frac{1}{1-a}}\right)$. Then there exist two strictly positive constants $\underline{l}, \bar{l}$ depending only on a, b and C such that*

$$\underline{l} \leq L \leq \bar{l}.$$

Proof. As there exists $Y \in \mathcal{Y}^*$ which satisfies $(A_{\frac{1}{1-a}})$, Y also satisfies (b_q^-) for every $q \in (0,1)$ by Lemma 2.2.6. Proposition 2.3.5 implies that $||L(1-a)||_\infty < +\infty$ and for every $\alpha > 0$, we have $\kappa_\alpha = \inf_{t\in[0,T]} L_t(-\alpha) > 0$. We infer from Proposition 2.4.7 that $L \leq \bar{l}$ where $\bar{l}$ is the root of the equation (2.23). Note that $\bar{l}$ depends only on b, C and $||L(1-a)||_\infty$. For the lower bound, we choose $\alpha < a$, $\beta \in (0, \frac{\alpha}{b})$ and apply Lemma 2.4.4 which yields for $t \in [0,T]$

$$\kappa_\alpha \leq \max\{C^\beta, C^{\frac{\alpha}{\beta}}\} L_t^\beta \left(1 + (1+\bar{l})^{\frac{\alpha}{a}-\beta}\right).$$

We deduce that $L \geq \underline{l}$ with $\underline{l}^\beta = \kappa_\alpha / \max\{C^\beta, C^{\frac{\alpha}{\beta}}\} \left(1 + (1+\bar{l})^{\frac{\alpha}{a}-\beta}\right)$. □

With Theorem 2.4.5, Proposition 2.4.7 and Theorem 2.4.9, we have reduced our analysis of integrability properties of L to that $L(p)$ for some $p \in (-\infty, 0) \cup (0,1)$. In Section 2.6, we will rely on this reduction to identify market models for which L is bounded away from 0 and ∞. A sufficient condition for L to be in $\mathcal{S}^r$ for some $r \geq 1$ will be given in Chapter 3.

2.5 Uniform bounds of L and integrability properties of the optimizers

Our goal in this section is to show how uniform bounds of L away from 0 and ∞ relate to the integrability properties of the optimizers: dual optimizer $\widehat{Y}$, optimal wealth $\widehat{X}$ and optimal trading strategy ν. We will rely on these properties in Section 2.7 to provide a stability analysis of the optimizers w.r.t. to misspecification in the risk preference and initial wealth.

2.5.1 Uniform bounds of L and reverse Hölder's condition for $\widehat{Y}$

In this section, we are interested in the uniform bounds of L and its links with the conditions (b_q^-) or (A_r) for $\widehat{Y}$ for some $q \in (0,1)$ or $r > 1$. These will complement Theorems 2.4.5 and 2.4.9 in terms of necessary conditions for L to be bounded away from 0 and ∞. We will distinguish the cases $a > 1$ and $a \leq 1$ which will be related respectively to the conditions (b_q^-) and (A_r) for $\widehat{Y}$.

Uniform bounds of L and the (b_q^-) condition for $\widehat{Y}$

We confine our attention in section to the class of utility functions U satisfying $(G_{a,b,C})$ with $a > 1$. The following theorem reduces the complete study of the boundedness of L away from 0 and ∞, and the condition (b_q^-) for $\widehat{Y}$ for any $q \in (0,1)$ to the condition $(b_{q'}^-)$ for any arbitrary element $Y \in \mathcal{Y}^*$ and for any $q' \in (0,1)$.

Theorem 2.5.1. *Suppose that U satisfies $(G_{a,b,C})$ with $a > 1$. Let $q \in (0,1)$. The following assertions are equivalent:*

i) L is bounded away from 0 and ∞.

ii) (b_q^-) holds for $\widehat{Y}$.

iii) (b_q^-) holds for some $Y \in \mathcal{Y}^*$.

iv) $L(p)$ is bounded away from 0 and ∞ for all $p < 0$.

Proof. i) $\Rightarrow$ ii) Let $\sigma \in \mathcal{T}(\mathbb{F})$. First we suppose that q satisfies $1 - \frac{1}{b} \leq q < 1$ and show that $\widehat{Y} = U'(\widehat{X})L$ satisfies $\left(b_q^-\right)$. By Proposition 2.4.2 there exists $C_1 > 0$ such that $L \leq C_1$. We infer from $(G_{a,b,C})$ that

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^q \middle| \mathcal{F}_\sigma\right] &= \frac{1}{L_\sigma^q}\mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^q 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] + \frac{1}{L_\sigma^q}\mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^q 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] \\
&\geq \frac{1}{C^q}\frac{1}{L_\sigma^q}\left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{qa} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] + \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{bq} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right). \quad (2.25)
\end{aligned}$$

For $p \in \left\{\frac{qa}{a-1}, \frac{qb}{b-1}\right\}$, the map $(0,+\infty) \ni y \mapsto y^p$ is convex as $p > 1$. By Jensen's inequality,

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{qa} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] &\geq \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{a-1} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{qa}{a-1}}, \\
\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{bq} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] &\geq \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{b-1} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{bq}{b-1}}.
\end{aligned}$$

With $\gamma = \frac{1}{C^q C_1}$, we deduce from (2.25) that

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^q \middle| \mathcal{F}_\sigma\right] \geq \gamma \left[\left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{a-1} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{qa}{a-1}} + \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{b-1} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{bq}{b-1}}\right].$$

As $\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{b-1} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] \in [0,1]$ and $\frac{bq}{b-1} \leq \frac{qa}{a-1}$, we have

$$\left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{b-1} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{bq}{b-1}} \geq \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{b-1} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{aq}{a-1}}.$$

Using the binomial inequalities $z_1^l + z_2^l \geq \frac{1}{2^l}(z_1 + z_2)^l$, $z_1, z_2 > 0, l > 0$, we obtain

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^q \middle| \mathcal{F}_\sigma\right] \geq \bar{\gamma} \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{a-1} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] + \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{b-1} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]\right)^{\frac{qa}{a-1}}, \tag{2.26}$$

where $\bar{\gamma} = \gamma 2^{-\frac{qa}{a-1}}$. It remains to show that the right hand term in (2.26) is bounded away from 0. For this, we use the fact that $\kappa = \operatorname{ess\,inf}_{\omega \in \Omega} \inf_{t \in [0,T]} L_t(\omega) > 0$ by i). The condition $(G_{a,b,C})$ gives

$$\begin{aligned} \kappa &\leq L_\sigma = \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] + \mathbb{E}\left[\frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] \\ &\leq C\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{b-1} 1_{\{\widehat{X}_T \leq \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right] + C\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{a-1} 1_{\{\widehat{X}_T > \widehat{X}_\sigma\}} \middle| \mathcal{F}_\sigma\right]. \end{aligned} \tag{2.27}$$

Inserting (2.27) in (2.26) yields $\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^q \middle| \mathcal{F}_\sigma\right] \geq \bar{\gamma}\left(\frac{\kappa}{C}\right)^{\frac{qa}{a-1}}$. We deduce that $\widehat{Y}$ satisfies (b_q^-). As $\widehat{Y}$ is a supermartingale, Lemma 2.2.6 entails that it satisfies (b_l^-) for all $l \in (0,1)$.
ii) $\Rightarrow$ iii). This is clear since $\widehat{Y} \in \mathcal{Y}^*$. iii) $\Rightarrow$ i) holds by Theorem 2.4.5. The equivalence between iii) and iv) follows from Proposition 2.3.5 and assertion ii) of Lemma 2.2.6. □

Theorem 2.5.1 extends Proposition 2.3.5 for power utility functions with risk aversion $1-p > 0$ to the larger class of utility functions U satisfying $(G_{a,b,C})$ with $a > 1$.

Remark 2.5.2. *Theorem 2.5.1 fails in general for $a \leq 1$.*

- *Consider the logarithmic utility $U(z) = \log z, z > 0$, which satisfies $(G_{1,1,1})$ and for which $L = 1$. For the continuous market model given in [KS99, Example 5.1], the corresponding dual optimizer is not uniformly integrable. Hence by Lemma 2.2.6, the dual optimizer does not satisfy the condition (b_q^-) for every $q \in (0,1)$. The assertion i) implies ii) in Theorem 2.5.1 fails.*

- *For $a \in (0,1)$, take $U(z) = \frac{z^{1-a}}{1-a}, z > 0$. In [FMW12, Proposition 5.1 and Theorem 5.2], market models are constructed for which the density of the minimal martingale measure satisfies the condition (b_q^-) for every $q \in (0,1)$. However $L(1-a)$ is unbounded. Therefore in Theorem 2.5.1, iii) does not imply i).*

Uniform bounds of L and the (A_r) condition for $\widehat{Y}$

We consider in this section the links between the boundedness of L and the condition (A_r) for $\widehat{Y}$. We recall that the condition (A_r) implies (b_q^-) for all $q \in (0,1)$. Thus building on Remark 2.5.2, we cannot expect the boundedness of L away from 0 and ∞ to be sufficient in all cases to guarantee that $\widehat{Y}$ satisfies (A_r) for some $r > 1$. Indeed, if $\widehat{Y}$ satisfies (A_r) for some $r > 1$, then by Proposition 2.3.5, $L(1/r)$ is bounded away from 0 and ∞, and the dual optimizer Y corresponding to the power utility case with risk aversion $1 - \frac{1}{r}$ satisfies as well (A_r). Thus in general for $\widehat{Y}$ to satisfy (A_r), it is necessary for an element $Y \in \mathcal{Y}^*$ to satisfy an analogue condition. The following theorem collects some sufficient conditions under which the boundedness of L away from 0 and ∞, leads to the condition (A_r) for $\widehat{Y}$ for some $r > 1$.

Theorem 2.5.3. *Suppose that U satisfies $(G_{a,b,C})$ with $0 < a \le b$. The following assertions hold:*

i) Assume that L is bounded away from 0 and ∞, and there exists $Y \in \mathcal{Y}^*$ satisfying (A_k) for some $k > 1$. Then $\widehat{Y}$ satisfies (A_r) for $r \ge 1 + bk$.

ii) Assume that L is bounded away from 0 and ∞, and $b < 1$. Then $\widehat{Y}$ satisfies $(A_{\frac{1}{1-b}})$.

Proof. i) Let $r \ge 1 + bk$ and $\sigma \in \mathcal{T}(\mathbb{F})$. As $\widehat{Y}_\sigma = U'(\widehat{X}_\sigma)L_\sigma$ and $L_T = 1$, we obtain

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] = L_\sigma^{\frac{1}{r-1}}\mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right]. \tag{2.28}$$

Using $(G_{a,b,C})$, we deduce that

$$\mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \le \mathbb{E}\left[\max\left\{1, C^{\frac{1}{r-1}}\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{b}{r-1}}\right\}\middle|\mathcal{F}_\sigma\right] \le \mathbb{E}\left[1 + C^{\frac{1}{r-1}}\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{b}{r-1}}\middle|\mathcal{F}_\sigma\right].$$

We set $p := \frac{b}{r-1}$. Clearly $p \le \frac{1}{k}$ and as Y satisfies (A_k), the opportunity process $L(p)$ is uniformly bounded by Proposition 2.3.5. It follows from the supermartingale property of $\widehat{X}^p L(p)$ that

$$\mathbb{E}\left[1 + C^{\frac{1}{r-1}}\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^p\middle|\mathcal{F}_\sigma\right] \le 1 + C^{\frac{1}{r-1}}L(p) \le 1 + C^{\frac{1}{r-1}}||L(p)||_\infty.$$

Since L is bounded from above, we deduce from (2.28) and the above inequalities that

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \le ||L||_\infty^{\frac{1}{r-1}}\left(1 + C^{\frac{1}{r-1}}||L(p)||_\infty\right) := K_r. \tag{2.29}$$

Hence $\widehat{Y}$ satisfies (A_r).

ii) We assume that $b < 1$ and L is bounded from above. Let $\sigma \in \mathcal{T}(\mathbb{F})$. With $r = \frac{1}{1-b}$, we have $\frac{1}{r-1} = \frac{1-b}{b}$. Therefore

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)L_\sigma}{U'(\widehat{X}_T)}\right)^{\frac{1-b}{b}}\middle|\mathcal{F}_\sigma\right] = L_\sigma^{\frac{1-b}{b}}\mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1-b}{b}}\middle|\mathcal{F}_\sigma\right]. \tag{2.30}$$

As U' is decreasing, $\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1-b}{b}} 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \le 1$. Now $(G_{a,b,C})$ implies that

$$\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1}{b}-1} 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} = \left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1}{b}} \frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)} 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \le C^{\frac{1}{b}} \frac{\widehat{X}_T U'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)}.$$

With the above inequalities, the definition of L and the monotonicity property of U' we have

$$\begin{aligned}\mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1-b}{b}}\middle|\mathcal{F}_\sigma\right] &\leq 1+\mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\frac{1-b}{b}}1_{\{\widehat{X}_\sigma\leq\widehat{X}_T\}}\middle|\mathcal{F}_\sigma\right]\\ &\leq 1+C^{\frac{1}{b}}\mathbb{E}\left[\frac{\widehat{X}_TU'(\widehat{X}_T)}{\widehat{X}_\sigma U'(\widehat{X}_\sigma)}\middle|\mathcal{F}_\sigma\right]=1+C^{\frac{1}{b}}L_\sigma.\end{aligned}$$

We infer from (2.30) that

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \leq ||L||_\infty^{\frac{1}{b}-1}+C^{\frac{1}{b}}||L||_\infty^{\frac{1}{b}}<\infty.$$

As σ is arbitrary, $\widehat{Y}$ satisfies (A_r). ☐

Remark 2.5.4. *Theorem 2.5.3 is stronger than the recent result [KW16, Theorem 3.4] which states that under the condition $(G_{a,b,C})$ with $a\in(0,1)$, $\widehat{Y}$ satisfies (A_r) with $r=1+\frac{b}{1-a}$ if there exits $Y\in\mathcal{Y}^*$ satisfying (A_k) with $k=\frac{1}{1-a}$. Indeed by Theorem 2.4.9, the existence of $Y\in\mathcal{Y}^*$ satisfying (A_k) with $k=\frac{1}{1-a}$ implies the boundedness of L away from 0 and ∞, and as a result all the conditions of Theorem 2.5.3 are satisfied and we recover that $\widehat{Y}$ satisfies (A_r) with $r=1+\frac{b}{1-a}$.*

The following proposition shows that in the special case $a=b\in(0,1)$, we have an equivalence between the boundedness of L away from 0 and ∞, and the condition (A_r) for $\widehat{Y}$ with $r=\frac{1}{1-a}$. It serves as a converse of Theorem 2.4.9 and extends Proposition 2.3.5 to the larger class of utility functions satisfying the condition $(G_{a,a,C})$. In practice, it will be useful to show that sharp conditions derived for the boundedness of $L(1-a)$ extend as well to L, see Remark 2.6.13.

Proposition 2.5.5. *Suppose that U satisfies $(G_{a,b,C})$ with $a=b\in(0,1)$. Let $r=\frac{1}{1-a}$. The following assertions are equivalent:*

A1) L is bounded away from 0 and ∞.

A2) (A_r) holds for $\widehat{Y}$.

A3) (A_r) holds for some $Y\in\mathcal{Y}^*$.

A4) $L(1-a)$ is bounded away from 0 and ∞.

Proof. A1) $\Rightarrow$ A2) follows from assertion ii) of Theorem 2.5.3 . A2) $\Rightarrow$ A3) holds since $\widehat{Y}\in\mathcal{Y}^*$. Observe that A3) $\Rightarrow$ A1) is a consequence of Theorem 2.4.9. By Proposition 2.3.5, A3) is equivalent to A4). ☐

For $a\geq 1$, a combination of Theorems 2.4.5 and 2.5.3 lead to the following sufficient condition for both the boundedness of L away from 0 and ∞, and the condition (A_r) for $\widehat{Y}$ for some $r>1$.

Proposition 2.5.6. *Suppose that U satisfies $(G_{a,b,C})$ with $a\geq 1$. Let $k>1$. Assume that (A_k) holds for some $Y\in\mathcal{Y}^*$. Then $\widehat{Y}$ satisfies (A_r) with $r=1+bk$ and L is bounded away from 0 and ∞.*

Proof. As $a\geq 1$ and (A_k) implies (b_q^-) for all $q\in(0,1)$, Theorem 2.4.5 entails that L is bounded away from 0. We infer from Theorem 2.5.3 that $\widehat{Y}$ satisfies (A_r) with $r=1+bk$. ☐

Observing carefully the proof of Theorem 2.5.3, the boundedness of L away from 0, is not needed. However, L bounded away from 0, is a necessary condition for $\widehat{Y}$ to satisfy the condition (A_r) for some $r > 1$ as the following lemma shows.

Lemma 2.5.7. *Suppose that U satisfies $(G_{a,b,C})$ with $a \in (0,1)$ and $\overline{u}_{1-a} < +\infty$. Assume that $\widehat{Y}$ satisfies (A_r) for some $r > 1$. Then there exists $\delta > 0$ depending only on a, C and r such that $L \geq \delta$.*

Proof. Let $\alpha > 0$ such that $\alpha b = \frac{1}{r}$ and $\sigma \in \mathcal{T}(\mathbb{F})$. The function $\mathbb{R} \ni x \mapsto x^{-\alpha}$ is decreasing and convex. Using Jensen's inequality and the supermartingale property of $\widehat{Y} = U'(\widehat{X})L$, we have

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^{-\alpha}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^{-\alpha}\frac{1}{L_\sigma^{-\alpha}}\middle|\mathcal{F}_\sigma\right] \geq 1.$$

Thus $L_\sigma^{-\alpha} \leq \mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^{-\alpha}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\alpha}\middle|\mathcal{F}_\sigma\right]$. An application of $(G_{a,b,C})$ yields

$$\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\alpha} \leq 1 + C^\alpha\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\alpha b} = 1 + C^\alpha\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{1}{r}}.$$

Since $\widehat{Y}$ satisfies (A_r), Proposition 2.3.5 entails that $L(1/r)$ is uniformly bounded. Employing the supermartingale property of the product $X^{\frac{1}{r}}L(1/r)$ and the previous inequalities, we obtain

$$L_\sigma^{-\alpha} \leq \mathbb{E}\left[\left(\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)}\right)^{\alpha}\middle|\mathcal{F}_\sigma\right] \leq 1 + C^\alpha\mathbb{E}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{1}{r}}\middle|\mathcal{F}_\sigma\right] \leq 1 + C^\alpha||L(1/r)||_\infty.$$

Take $\delta = \left(\frac{1}{1+C^\alpha||L(1/r)||_\infty}\right)^{\frac{1}{\alpha}}$. □

In view of Proposition 2.5.5 and Lemma 2.5.7, the boundedness of L away from 0 and ∞ does not only lead to a sufficient condition for $\widehat{Y}$ to satisfy (A_r) for some $r > 1$, but also provides a necessary condition. A particular benefit for addressing the condition (A_r) for $\widehat{Y}$ by looking at the boundedness of L is the simple verification principle given by Proposition 2.5.6 for the case $a \geq 1$. This is treated in [KW16, Corollary 5.5] only for S with continuous paths and for a continuous filtration $\mathbb{F}$. In Remark 2.5.14, we will highlight the importance of the boundedness of L in conferring to $\widehat{Y}$ the structure of the stochastic exponential of a BMO martingale in a general setting.

2.5.2 Uniform bounds of L and moments of the optimal wealth process

So far we have illustrated the importance of L in yielding necessary and sufficient conditions for $\widehat{Y}$ to satisfy the condition (b_q^-) for some $q \in (-\infty, 0) \cup (0,1)$. In this section, we turn our attention to the optimal terminal wealth $\widehat{X}_T$. Contrary to the dual optimizer $\widehat{Y}$ which is directly connected to L via the relation $\widehat{Y} = U'(\widehat{X})L$, we do not have such a connection for the optimal wealth process $\widehat{X}$. A priori, it is not clear if the boundedness of L away from 0 and ∞, ensures that $\widehat{X}_T$ admits moments of a certain order. For this reason, we will work under the assumption that the *dual optimal martingale measure exists*, i.e. $\widehat{Y}/\widehat{Y}_0$ is the density process of an equivalent local martingale measure $\widehat{\mathbb{Q}}$, and look for the moments of $\widehat{X}_T$ w.r.t. the measure $\widehat{\mathbb{Q}}$. Now observe that if $\widehat{\mathbb{Q}}$ is well defined, then for $\gamma > 1$ and $\sigma \in \mathcal{T}(\mathbb{F})$, we have

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\gamma}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\gamma}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{\widehat{X}_T\widehat{Y}_T}{\widehat{X}_\sigma\widehat{Y}_\sigma}\right)^{\gamma}\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\gamma-1}\middle|\mathcal{F}_\sigma\right].$$

Thus if $\widehat{X}\widehat{Y}$ satisfies (R_p) for some $p>1$ and $\widehat{Y}$ satisfies (A_r) for some $r>1$, then for γ suitably chosen and applying Hölder's inequality, $\widehat{X}$ will satisfy (R_γ) w.r.t. to the measure $\widehat{\mathbb{Q}}$ and therefore we will have $\widehat{X}_T \in L^\gamma(\widehat{\mathbb{Q}})$. The process L was defined via the martingale $\widehat{X}\widehat{Y}$ and we expect the boundedness of L to be connected to the condition (R_p) for $\widehat{X}\widehat{Y}$. We will proceed in two steps to infer moments of $\widehat{X}_T$ from the boundedness of L. In a first step, we establish the equivalence between the boundedness of L away from 0 and ∞, and the condition (A_r) stated for $\widehat{X}\widehat{Y}$ for some $r>1$. In a second step, we provide a sufficient condition for the existence of $\widehat{Y}/\widehat{Y}_0$ to define the density process of an equivalent local martingale measure for S and how to choose γ.

Uniform bounds of L and the (A_r) condition for $\widehat{X}\widehat{Y}$

Before stating the main result of this section, we recall some facts and properties of the martingale $\widehat{X}\widehat{Y}$. We consider the process $S^{\widehat{X}}$ and the probability measure $\mathbb{Q}^{\widehat{X}}$ defined by

$$S^{\widehat{X}} = \left(\frac{1}{\widehat{X}}, \frac{S}{\widehat{X}}\right) \text{ and } \left.\frac{d\mathbb{Q}^{\widehat{X}}}{d\mathbb{P}}\right|_{\mathcal{F}_T} = \frac{\widehat{X}_T\widehat{Y}_T}{xy}. \tag{2.31}$$

Clearly $S^{\widehat{X}}$ is the asset price process discounted by the numeraire $\widehat{X}$. It is well known that $\widetilde{X}$ is a wealth process from trading in the assets with price process $S^{\widehat{X}}$ if and only if $\widetilde{X}\widehat{X}$ is a wealth process from trading in the assets with prices modeled by S, see [DS95, Theorem 11]. Moreover, $\mathbb{Q}^{\widehat{X}}$ is an equivalent local martingale measure for $S^{\widehat{X}}$, see [DS94, Theorem 5.6].
The following proposition shows that $(A_{r'})$ holds for $\widehat{Y}$ for some $r'>1$ if (A_r) holds for $\widehat{X}\widehat{Y}$ for some $r>1$. It does not depend on the condition $(G_{a,b,C})$ nor L. The process $\widehat{X}\widehat{Y}$ is thus an auxiliary tool to investigate the Muckenhoupt's condition for $\widehat{Y}$.

Proposition 2.5.8. *Assume that there exists $Y \in \mathcal{Y}^*$ satisfying (A_k) for some $k>1$ and $\widehat{X}\widehat{Y}$ satisfies (A_r) for some $r>1$. Then $\widehat{Y}$ satisfies (A_γ) for some $\gamma>1$ and in addition there exists a constant $\delta>0$ such that for every $\sigma \in \mathcal{T}(\mathbb{F})$, we have*

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\frac{1}{r}}\middle|\mathcal{F}_\sigma\right] \le \delta. \tag{2.32}$$

Proof. Let $q>1, \bar{q}=\frac{q}{q-1}$ and $\gamma = 1+q(r-1)$. Let $\sigma \in \mathcal{T}(\mathbb{F})$. By Hölder's inequality one has

$$\begin{aligned}\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{\gamma-1}}\middle|\mathcal{F}_\sigma\right] &= \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma\widehat{Y}_\sigma}{\widehat{X}_T\widehat{Y}_T}\right)^{\frac{1}{\gamma-1}}\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{1}{\gamma-1}}\middle|\mathcal{F}_\sigma\right] \\ &\le \left(\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma\widehat{Y}_\sigma}{\widehat{X}_T\widehat{Y}_T}\right)^{\frac{q}{\gamma-1}}\middle|\mathcal{F}_\sigma\right]\right)^{\frac{1}{q}}\left(\mathbb{E}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{\bar{q}}{\gamma-1}}\middle|\mathcal{F}_\sigma\right]\right)^{\frac{1}{\bar{q}}}.\end{aligned}$$

Note that $\frac{q}{\gamma-1} = \frac{1}{r-1}$ and $\frac{\bar{q}}{\gamma-1} = \frac{1}{(q-1)(r-1)}$. By assumption, there exists $C_r>0$ such that $\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma\widehat{Y}_\sigma}{\widehat{X}_T\widehat{Y}_T}\right)^{\frac{q}{\gamma-1}}\middle|\mathcal{F}_\sigma\right] \le C_r$. Choosing q such that $(q-1)(r-1)\ge k$, we have $\frac{\bar{q}}{\gamma-1} \le \frac{1}{k} \in (0,1)$. We set $p=\frac{\bar{q}}{\gamma-1}$. Since there exists $Y\in\mathcal{Y}^*$ which satisfies (A_k), we deduce from Proposition 2.3.5 that $L(p)$ is bounded. Using the supermartingale property of the product $X^pL(p)$, we have

$$\mathbb{E}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{\bar{q}}{\gamma-1}}\middle|\mathcal{F}_\sigma\right] \le L_\sigma(p) \le ||L(p)||_\infty.$$

Altogether, it follows that $\widehat{Y}$ satisfies (A_γ) with $\gamma = 1 + q(r-1), q > 1 + \frac{k}{r-1}$.

Next we prove (2.32). Let $p = \frac{1}{r} \in (0,1)$. We denote by $\overline{u}_p^{S^{\widehat{X}}}$ (resp. $\overline{v}_p^{S^{\widehat{X}}}$) the value function (resp. dual value function) for the power utility maximization problem with relative risk aversion $1-p$ and underlying asset price $S^{\widehat{X}}$. Since $\mathbb{Q}^{\widehat{X}} \in \mathcal{M}^e(S^{\widehat{X}})$ and $\widehat{X}\widehat{Y}$ satisfies $(A_{\frac{1}{p}})$, we deduce from Remark 2.3.1 that $v_p^{S^{\widehat{X}}}(z) < +\infty$, $\forall z > 0$. Thus $\overline{u}_p^{S^{\widehat{X}}}(z) < +\infty$, $\forall z > 0$ by [KS03, Theorem 2]. From Proposition 2.3.2, we infer that the corresponding opportunity process $L^{S^{\widehat{X}}}(p)$ is well defined. By Proposition 2.3.5, $L^{S^{\widehat{X}}}(p)$ is bounded since $\widehat{X}\widehat{Y}$ satisfies $(A_{\frac{1}{p}})$. As $\frac{1}{\widehat{X}}$ is a wealth process from trading into the assets with price process $S^{\widehat{X}}$, $\left(\frac{1}{\widehat{X}}\right)^p L^{S^{\widehat{X}}}(p)$ is a supermartingale. Hence for $\sigma \in \mathcal{T}(\mathbb{F})$ we have

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\frac{1}{r}}\Big|\mathcal{F}_\sigma\right] \leq L_\sigma^{S^{\widehat{X}}}(p) \leq ||L^{S^{\widehat{X}}}(p)||_\infty.$$

□

Remark 2.5.9. *For logarithmic utility $U(z) = \log z, z > 0$, we have $\widehat{X}\widehat{Y} = 1$. By Remark 2.5.2, the dual optimizer might fail to be uniformly integrable and thus will not satisfy a Muckenhoupt's condition. Hence, the existence of $Y \in \mathcal{Y}^*$ satisfying (A_k) in Proposition 2.5.8 is necessary.*

The following theorem shows that the condition (A_r) for $\widehat{X}\widehat{Y}$ is equivalent to the boundedness of L away from 0 and ∞, provided (A_k) holds for some $Y \in \mathcal{Y}^*$.

Theorem 2.5.10. *Suppose that U satisfies $(G_{a,b,C})$ with $0 < a \leq b$. Assume that there exists $Y \in \mathcal{Y}^*$ satisfying (A_k) for some $k > 1$. The following assertions are equivalent:*

i) $\widehat{X}\widehat{Y}$ satisfies (A_r) for $r > \max\left\{\frac{1+a}{a}, 1 + bk\right\}$.

ii) L is bounded away from 0 and ∞.

Proof. We first prove that i) implies ii). For this purpose, let $r > 1$ and let us first prove that L is bounded away from 0. As $\widehat{X}\widehat{Y}$ satisfies (A_r), we infer from Proposition 2.5.8 that $\widehat{Y}$ satisfies $(A_{r'})$ for some $r' > 1$ and from Lemma 2.5.7 that L is bounded away from 0. Next we show that L is bounded from above. Let $\gamma \in (0,1)$ with $\gamma b < \frac{1}{r}$. Since $\widehat{Y}$ satisfies $(A_{r'})$, it satisfies (b_γ^-) by Lemma 2.2.6. Therefore there exists K_γ such that for every $\sigma \in \mathcal{T}(\mathbb{F})$, we have

$$K_\gamma \leq \mathbb{E}\left[\left(\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\right)^\gamma\Big|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)L_\sigma}\right)^\gamma\Big|\mathcal{F}_\sigma\right].$$

Let $\sigma \in \mathcal{T}(\mathbb{F})$. Using $(G_{a,b,C})$, we have $\left(\frac{U'(\widehat{X}_T)}{U'(\widehat{X}_\sigma)}\right)^\gamma \leq \max\left\{1, C^\gamma\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\gamma b}\right\} \leq 1 + C^\gamma\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\gamma b}$. Hence

$$K_\gamma L_\sigma^\gamma \leq 1 + C^\gamma \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^{\gamma b}\Big|\mathcal{F}_\sigma\right]. \tag{2.33}$$

As $\widehat{X}\widehat{Y}$ satisfies (A_r) and $r < \frac{1}{\gamma b}$, it also satisfies $\left(A_{\frac{1}{\gamma b}}\right)$. Proposition 2.5.8 ensures that the right hand term of (2.33) is uniformly bounded and therefore L is bounded from above.

We now show that ii) implies i). Choose $r > \max\left\{\frac{1+a}{a}, 1+bk\right\}$, $\alpha = \frac{1}{r-1}$ and $\sigma \in \mathcal{T}(\mathbb{F})$. Then

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma \widehat{Y}_\sigma}{\widehat{X}_T \widehat{Y}_T}\right)^\alpha \middle| \mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma \widehat{Y}_\sigma}{\widehat{X}_T \widehat{Y}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] + \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma \widehat{Y}_\sigma}{\widehat{X}_T \widehat{Y}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right].$$

L being bounded and $r > 1+bk$, $\widehat{Y}$ satisfies (A_r) by Theorem 2.5.3. More precisely, we have

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^\alpha \middle| \mathcal{F}_\sigma\right] \le K_r,$$

where K_r is the constant given by (2.29). Hence

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma \widehat{Y}_\sigma}{\widehat{X}_T \widehat{Y}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \le \mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma \le \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \le K_r. \tag{2.34}$$

The function U' is decreasing. Thus $\frac{U'(\widehat{X}_\sigma)}{U'(\widehat{X}_T)} 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \le 1$. Since $\widehat{Y} = U'(\widehat{X})L$, we have

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma \widehat{Y}_\sigma}{\widehat{X}_T \widehat{Y}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \le L_\sigma^\alpha \mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha \middle| \mathcal{F}_\sigma\right]. \tag{2.35}$$

The inequality $r > 1 + \frac{1}{a}$ implies that $\alpha = \frac{1}{r-1} < a$. Choosing $\beta \in (0, \frac{\alpha}{b})$, Lemma 2.4.4 implies that

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma}{\widehat{X}_T}\right)^\alpha \middle| \mathcal{F}_\sigma\right] \le C^\beta L_\sigma^\beta + C^{\frac{\alpha}{a}} L_\sigma^{\frac{\alpha}{a}} \le \max\{C^\beta, C^{\frac{\alpha}{a}}\} ||L||_\infty^\beta \left(1 + ||L||_\infty^{\frac{\alpha}{a}-\beta}\right).$$

We infer from (2.35) that

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma \widehat{Y}_\sigma}{\widehat{X}_T \widehat{Y}_T}\right)^\alpha 1_{\{\widehat{X}_\sigma > \widehat{X}_T\}} \middle| \mathcal{F}_\sigma\right] \le \max\{C^\beta, C^{\frac{\alpha}{a}}\} ||L||_\infty^{\beta+\alpha} \left(1 + ||L||_\infty^{\frac{\alpha}{a}-\beta}\right).$$

Combining (2.34) and (2.35) we obtain

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma \widehat{Y}_\sigma}{\widehat{X}_T \widehat{Y}_T}\right)^\alpha \middle| \mathcal{F}_\sigma\right] \le K_r + \max\{C^\beta, C^{\frac{\alpha}{a}}\} ||L||_\infty^{\beta+\alpha} \left(1 + ||L||_\infty^{\frac{\alpha}{a}-\beta}\right). \tag{2.36}$$

Since σ is arbitrary, we deduce that $\widehat{X}\widehat{Y}$ satisfies (A_r). The proof is complete. □

Moments of the optimal wealth and BMO representation of the dual optimizer

We will now apply Theorem 2.5.10 to show that the boundedness of L away from 0 and ∞ enhances the moments of $\widehat{X}$ w.r.t. the dual optimal martingale measure. In Section 2.6.2, we will show that for continuous asset prices, a similar result holds with the *minimal martingale measure*. We recall that $\widehat{X} = x\mathcal{E}(\nu \cdot R)$ where ν is the optimal trading strategy and $R = \int_0^\cdot \frac{1}{S_-} dS$.

Theorem 2.5.11. *Suppose that U satisfies $(G_{a,b,C})$ with $a \le b$, and there exists $Y \in \mathcal{Y}^*$ satisfying (A_k) for some $k > 1$. Let ν be the optimal trading strategy and $\widehat{\mathbb{Q}}$ be the probability measure equivalent to $\mathbb{P}$ with density*

$$d\widehat{\mathbb{Q}}/d\mathbb{P}|_{\mathcal{F}_T} = \widehat{Y}_T/\widehat{Y}_0. \tag{2.37}$$

Suppose additionally that $\widehat{X}$ satisfies (J), the process $\widehat{Y}$ is a local martingale and L is bounded away from 0 and ∞. Then the following hold:

a) $\widehat{\mathbb{Q}} \in \mathcal{M}^e(S)$,

b) The process $\nu \cdot R \in BMO(\widehat{\mathbb{Q}})$ *and there exists* $\gamma > 1$ *such that* $\widehat{X} \in \mathcal{S}^\gamma(\widehat{\mathbb{Q}})$.

c) There exists a BMO martingale $\widetilde{M}$ *such that* $\widehat{Y} = \widehat{Y}_0\mathcal{E}(\widetilde{M})$.

If $a > 1$, *then the existence of* Y *satisfying* (A_k) *for some* $k > 1$ *can be omitted.*

For the proof, we need the following lemma.

Lemma 2.5.12. *Suppose that* U *satisfies* $(G_{a,b,C})$ *with* $a \leq b$. *Assume that* $\widehat{X}$ *and* L *satisfy the condition* (J). *Then:*

i) $\widehat{Y}$ *and* $\widehat{X}\widehat{Y}$ *satisfy the condition* (J).

ii) If $\widehat{Y}$ *is a local martingale satisfying* (b_q^-) *for some* $q \in (0,1)$, *then there exists a BMO martigale* $\widetilde{M}$ *such that* $\widehat{Y} = \widehat{Y}_0\mathcal{E}(\widetilde{M})$.

Proof. i) Let $k > 1$ and $l > 1$ such that

$$\frac{1}{k} \leq \frac{\widehat{X}}{\widehat{X}_-} \leq k \text{ and } \frac{1}{l} \leq \frac{L}{L_-} \leq l. \tag{2.38}$$

Note that as $k > 1$ and $a \leq b$, we have $k^a \leq k^{\mathrm{b}}$. Using the condition $(G_{a,b,C})$ and (2.38), we have

$$\begin{aligned}
\frac{U'(\widehat{X})}{U'(\widehat{X}_-)} &\leq C\left(\frac{\widehat{X}_-}{\widehat{X}}\right)^{\mathrm{b}} 1_{\{\widehat{X}\leq\widehat{X}_-\}} + C\left(\frac{\widehat{X}_-}{\widehat{X}}\right)^{a} 1_{\{\widehat{X}>\widehat{X}_-\}} \leq Ck^{\mathrm{b}}, \\
\frac{U'(\widehat{X})}{U'(\widehat{X}_-)} &\geq \frac{1}{C}k^{-a}1_{\{\widehat{X}\leq\widehat{X}_-\}} + \frac{1}{C}k^{-b}1_{\{\widehat{X}>\widehat{X}_-\}} \geq C^{-1}k^{-b}.
\end{aligned}$$

As $\widehat{X}$, $U'(\widehat{X})$ and L satisfy the condition (J), one verifies that $\widehat{Y} = U'(\widehat{X})L$ and $\widehat{X}\widehat{Y} = \widehat{X}U'(\widehat{X})L$ satisfy as well the condition (J).
ii) As $\widehat{Y}$ is a strictly positive local martingale, we have $\widehat{Y} = \widehat{Y}_0\mathcal{E}(\widetilde{M})$ where $\widetilde{M} = \int_0^\cdot \frac{1}{\widehat{Y}_-}d\widehat{Y}$. The process $\widehat{Y}$ being a supermartingale, it satisfies $\left(b_1^+\right)$. By hypothesis and the first assertion, $\widehat{Y}$ satisfies $\left(b_q^-\right)$ and (J). Hence, by Gehring's lemma [DDM79, Proposition 4] there exists $\epsilon > 0$ such that $\widehat{Y}$ satisfies $\left(b_{1+\epsilon}^+\right)$. We infer from Proposition 2.2.7 that $\widetilde{M}$ is a BMO martingale. □

Proof of Theorem 2.5.11. a) By hypothesis, there exists $Y \in \mathcal{Y}^*$ satisfying (A_k) and $\widehat{Y}$ is a local martingale. Due to the boundedness of L, Theorem 2.5.3 implies that $\widehat{Y}$ satisfies (A_r) with $r = 1+bk$ and therefore it is a uniformly integrable martingale. Hence $\widehat{Y}/\widehat{Y}_0 = \left(\mathbb{E}\left[d\widehat{\mathbb{Q}}/d\mathbb{P}|\mathcal{F}_t\right]\right)_{t\in[0,T]}$ is the density process of the measure $\widehat{\mathbb{Q}}$. Since $\widehat{Y} \in \mathcal{Y}$, we deduce from [DS94, Theorem 5.6] that $\widehat{\mathbb{Q}} \in \mathcal{M}^e(S)$.
b) First note that as $\widehat{X}\widehat{Y}$ is a uniformly integrable martingale (see Theorem 2.2.12), $\widehat{X}$ is a $\widehat{\mathbb{Q}}$-martingale. Observe that $\widehat{X} = x\mathcal{E}(\nu \cdot R)$ is a $\widehat{\mathbb{Q}}$-local martingale satisfying the condition (J). So to prove the assertion, it suffices to show that $\widehat{X}$ satisfies (R_γ) under the measure $\widehat{\mathbb{Q}}$ for some $\gamma > 1$ and then apply successively Proposition 2.2.7 and Corollary 2.2.8. To achieve this, we use the fact that there exists $p > 1$ such that $\widehat{X}\widehat{Y}$ satisfies (R_p). Indeed by Theorem 2.5.10 and Lemma 2.5.12, $\widehat{X}\widehat{Y}$ satisfies (A_r) for some $r > 1$ and the condition (J). Thus by Gehring's lemma [DDM79, Proposition 4], there exists $p > 1$ such that $\widehat{X}\widehat{Y}$ satisfies (R_p). Let $\gamma, \alpha > 1$

and such that $\gamma\alpha = p$ to be made precise later. Let $\sigma \in \mathcal{T}(\mathbb{F})$. Using the fact that $\widehat{\mathbb{Q}}$ has density process $\widehat{Y}/\widehat{Y}_0$ and Hölder's inequality, we obtain

$$\begin{aligned}\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\gamma}\Big|\mathcal{F}_\sigma\right] &= \mathbb{E}\left[\frac{\widehat{Y}_T}{\widehat{Y}_\sigma}\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\gamma}\Big|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{\widehat{X}_T\widehat{Y}_T}{\widehat{X}_\sigma\widehat{Y}_\sigma}\right)^{\gamma}\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\gamma-1}\Big|\mathcal{F}_\sigma\right]\\ &\le \left(\mathbb{E}\left[\left(\frac{\widehat{X}_T\widehat{Y}_T}{\widehat{X}_\sigma\widehat{Y}_\sigma}\right)^{\gamma\alpha}\Big|\mathcal{F}_\sigma\right]\right)^{\frac{1}{\alpha}}\left(\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{(\gamma-1)\alpha}{\alpha-1}}\Big|\mathcal{F}_\sigma\right]\right)^{\frac{\alpha-1}{\alpha}}.\end{aligned}$$

We take $\gamma = p\frac{1+kb}{1+kbp}$ and $\alpha = \frac{1+kpb}{1+kb}$. Then $\gamma\alpha = p$, $\frac{\alpha}{\alpha-1} = \frac{1+kpb}{kb(p-1)}$ and $\frac{(\gamma-1)\alpha}{\alpha-1} = \frac{1}{kb}$. As $\widehat{X}\widehat{Y}$ satisfies (R_p), there exists a constant Θ independent of σ such that

$$\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\gamma}\Big|\mathcal{F}_\sigma\right] \le \Theta\left(\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{kb}}\Big|\mathcal{F}_\sigma\right]\right)^{\frac{\alpha-1}{\alpha}}.$$

The condition $(G_{a,b,C})$ and the supermartingale property of the product $\widehat{X}^{\frac{1}{k}}L(\frac{1}{k})$ yield

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{kb}}\Big|\mathcal{F}_\sigma\right] \le L_\sigma^{\frac{1}{kb}} + C^{\frac{1}{kb}}L_\sigma^{\frac{1}{kb}}\mathbb{E}\left[\left(\frac{\widehat{X}_T}{\widehat{X}_\sigma}\right)^{\frac{1}{k}}\Big|\mathcal{F}_\sigma\right] \le ||L||_\infty^{\frac{1}{kp}}(1 + C^{\frac{1}{kb}}L_\sigma(1/k)).$$

Now by hypothesis, there exists $Y \in \mathcal{Y}^*$ satisfying (A_k) and we infer from Proposition 2.3.5 that $||L(1/k)||_\infty < +\infty$. We deduce from the above two estimates that $\widehat{X}$ satisfies (R_γ) under the measure $\widehat{\mathbb{Q}}$. We obtain the assertion by applying Proposition 2.2.7 and Corollary 2.2.8.
c)By Theorem 2.5.3, $\widehat{Y}$ satisfies (A_r) with $r = 1 + bk$. Hence it satisfies (b_q^-) for all $q \in (0,1)$ by Lemma 2.2.6. Assertion c) is a consequence of Lemma 2.5.12.

Assume that $a > 1$. Then $\widehat{Y}$ satisfies (b_q^-) for all $q \in (0,1)$ by Theorem 2.5.1. Applying similar arguments as in Lemma 2.5.12, $\widehat{Y}$ satisfies the condition (J) and $(b_{1+\epsilon}^+)$ for some $\epsilon > 0$. We deduce from Proposition 2.2.7 that $\widehat{Y}$ satisfies the condition (A_k) for some $k > 1$. We can therefore repeat the same arguments as before to show a), b) and c). □

Remark 2.5.13. *The condition (J) for $\widehat{X}\widehat{Y}$ and/or $\widehat{X}$ is the most restrictive assumption in Theorem 2.5.11 for which we do not have a verification principle for a general semimartingale model S. Regarding the local martingale property of $\widehat{Y}$, we refer to the recent work [KW16, Proposition 4.2] which gives sufficient conditions. For models S with continuous paths, $\widehat{Y}$ is always a local martingale, see [KW16, Lv07]. Moreover , $\widehat{X}$ is continuous and therefore satisfies condition (J).*

Remark 2.5.14. *Assertion c) in Theorem 2.5.11 generalizes [KW16, Corollaries 5.5 and 5.6] which assumes the filtration $\mathbb{F}$ to be continuous. In such a filtration the condition (J) is satisfied since $\widehat{Y}$ is a continuous local martingale. If the filtration $\mathbb{F}$ fails to be continuous, then $\widehat{Y}$ might be discontinuous and in this case, the additional condition (J) is necessary for a positive uniformly integrable martingale to be written as the stochastic exponential of a BMO martingale , see [DDM79, ISS79]. As $\widehat{X}$ is continuous, the condition (J) for $\widehat{Y}$ is precisely equivalent to $\frac{L}{L_-}$ being bounded away from 0 and ∞. The latter equivalence enlightens the role played by the process L to endow $\widehat{Y}$ with nice representation properties in a general filtration.*

2.6 Market models and the boundedness of the opportunity process

The results in Sections 2.4 and 2.5 relate the boundedness of L away from 0 and ∞, to the condition $(b_q^-), q \in (-\infty, 0) \cup (0, 1)$ for some element $Y \in \mathcal{Y}^*$. For some models of the price process $S = \mathcal{E}(R)$ such as the stochastic volatility models of Heston [Hes93] or Barndorff-Nielsen-Shephard [BNS01], the process L is not bounded away from 0 and ∞. Indeed in these cases, the opportunity process for power utility $L(p)$ is an exponentially affine process of the corresponding volatility process which is unbounded[4] (see [KMK10, Section 3]) and therefore $L(p)$ is not bounded away from 0 and ∞. Thus for U satisfying $(G_{a,b,C})$, one infers from Theorem 2.5.1 that L is not bounded away from 0 and ∞ for $a > 1$. This is also the case for $b < 1$ by Theorem 2.5.3.

Our aim in this section is to give precise sharp conditions for the validity of the boundedness of L away from 0 and ∞ in market models of exponential Lévy types and continuous price processes. Since we do not have an explicit expression of L, we will make use of Theorems 2.4.5 and 2.4.9 relating the boundedness of L away from 0 and ∞, to that of $L(p)$ for some $p \in (-\infty, 0) \cup (0, 1)$.

2.6.1 Exponential Lévy models

We recall that $S = S_0\mathcal{E}(R)$ where R is an $\mathbb{R}^n$-valued semimartingale with $R_0 = 0$. We suppose that

Assumption 2.6.1. *R is a Lévy process[5] and S is positive.*

Under Assumption 2.6.1, the problem (2.4) has been investigated for the logarithmic utility function [Kal00] and the class of power utilities $U(z) = \frac{z^p}{p}, z > 0, p \in (-\infty, 0) \cup (0, 1)$ in [Nut12b]. A particular feature of the opportunity process $L(p)$ in this context is that it is a deterministic function of time. More precisely, it has the following form $L_t(p) = \exp(\alpha(1-p)(T-t)), t \in [0, T]$ where α is a constant depending on p and the Lévy triplet $\left(b^R, c^R, F^R\right)$ of R (see [Nut12a, Theorem 3.2]). $L(p)$ is therefore uniformly bounded away from 0 and ∞ for $p \in (-\infty, 0) \cup (0, 1)$ provided the value function $\overline{u}_p$ is finite. Theorems 2.4.5 and 2.4.9 lead to the following result concerning the boundedness of L.

Proposition 2.6.2. *Suppose that Assumption 2.6.1 holds and U satisfies $(G_{a,b,C})$ with $a \leq b$. Assume either that $a \neq 1$ and $\overline{u}_{1-a} < +\infty$, or $a = 1$ and $\overline{u}_p < +\infty$ for some $p \in (0, 1)$. Then*

1. *L is bounded away from 0 and ∞,*
2. *For $a \leq 1$, $\widehat{Y}$ satisfies (A_r) for some $r > 1$ and for $a > 1$, $\widehat{Y}$ satisfies (b_q^-) for all $q \in (0, 1)$.*

Proof. We consider the case $a \neq 1$ and $\overline{u}_{1-a} < +\infty$. By Theorem 3.2 in [Nut12b], $L(1-a)$ is bounded away from 0 and ∞. For $a > 1$, Theorem 2.5.1 implies that L is bounded away from 0 and ∞, and $\widehat{Y}$ satisfies (b_q^-) for some $q \in (0, 1)$. For $a < 1$, we infer from Proposition 2.3.5 and Theorem 2.4.9 that there exists $Y \in \mathcal{Y}^*$ which satisfies $(A_{\frac{1}{1-a}})$ and L is bounded away from 0 and ∞. Theorem 2.5.3 implies that $\widehat{Y}$ satisfies (A_r) with $r = 1 + \frac{b}{1-a}$.
We now assume that $a = 1$ and there exists $p \in (0, 1)$ such that $\overline{u}_p < +\infty$. Then $L(p)$ is bounded away from 0 and ∞ by Theorem 3.2 in [Nut12b]. Proposition 2.3.5 entails that there

[4] For the Heston model, the volatility process is the square root process while for the Barndorff Nielsen and Shephard model, it is a Lévy driven Ornstein-Uhlenbeck process

[5] A Lévy process is a càdlàg process having stationary and independent increments, and stochastically continuous, see [Sat99]

exists $Y \in \mathcal{Y}^*$ which satisfies $(A_{\frac{1}{p}})$. As $a = 1$, Theorem 2.4.5 and Proposition 2.5.6 guarantee that L is bounded away from 0 and ∞, and $\widehat{Y}$ satisfies (A_r) with $r = 1 + b\frac{1}{p}$. □

One deduces from Proposition 2.6.2 that for exponential Lévy models, the dual optimizer $\widehat{Y}$ is a true martingale if it possesses the local martingale property.

Corollary 2.6.3. *In addition to the assumptions of Proposition 2.6.2, suppose that $\widehat{Y}$ is a local martingale. Then the measure $\widehat{\mathbb{Q}}$ defined by $\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}} = \frac{\widehat{Y}_T}{\widehat{Y}_0}$ is an equivalent local martingale measure for S.*

Remark 2.6.4. *A necessary and sufficient condition for $\overline{u}_p, p \in (0,1)$ to be finite is given by [Nut12b, Corollary 3.7] in terms of the Lévy measure F^R of R.*

2.6.2 Case for continuous asset price processes

In this section, we consider the particular case where the risky asset price process S is a continuous semimartingale. We provide concrete conditions on its finite variation part that pertain to ensuring that the resulting opportunity process L be bounded away from 0 and ∞. So we will work under the following standing assumption:

Assumption 2.6.5. *The asset price process S is continuous.*

As $S = S_0\mathcal{E}(R)$ is continuous, the condition $\mathcal{M}^e(S) \neq \emptyset$ entails that it satisfies the structure condition (see [Sch95]), i.e. there exists an $\mathbb{R}^n$-valued continuous martingale M starting at 0 and $\mu \in \mathcal{L}(M)$ such that for $t \in [0,T]$

$$R_t \;=\; M_t + \int_0^t d\langle M\rangle_u \mu_u. \tag{2.39}$$

Let Z^μ be the process defined by $Z^\mu = \mathcal{E}\left(-\mu \cdot M\right)$ and $\mathbb{Q}^\mu$ the probability measure defined by

$$\frac{d\mathbb{Q}^\mu}{d\mathbb{P}}\bigg|\mathcal{F}_T = Z^\mu_T. \tag{2.40}$$

Note that $Z^\mu S$ is a local martingale and $Z^\mu \in \mathcal{Y}^*$. If $\mu \cdot M \in BMO$, then Z^μ is a uniformly integrable martingale by [Kaz94, Theorem 2.3]. Moreover, $\mathbb{Q}^\mu \in \mathcal{M}^e(S)$ and $\mathbb{Q}^\mu$ is referred to as the *minimal martingale measure*, see [Sch95]. As $Z^\mu \in \mathcal{Y}^*$, L is bounded away from 0 and ∞, if Z^μ satisfies (A_r) for some $r > 1$ appropriately chosen depending on whether $a \geq 1$ or $a \in (0,1)$. The local martingale $\mu \cdot M$ being continuous, the condition (A_r) for some $r > 1$ stated for Z^μ entails a BMO property for $\mu \cdot M$. The following lemma shows that such a property is actually necessary for the boundedness of L.

Lemma 2.6.6. *Suppose that U satisfies $(G_{a,b,C})$ and L is bounded away from 0 and ∞. Then $\mu \cdot M$ is a BMO martingale if $a > 1$ or $b < 1$.*

Proof. Since $a > 1$ or $b < 1$, and L is bounded away from 0 and ∞, we infer from Theorems 2.5.1, 2.5.3 and Lemma 2.2.6 that $\widehat{Y}$ satisfies $\left(b_q^-\right)$ for all $q \in (0,1)$. As S is continuous, $\widehat{X}$ is continuous. Moreover, it follows from [Lv07, Proposition 3.2 and Corollary 3.3] that $\widehat{Y}$ is a local martingale of the form $\widehat{Y}_0\mathcal{E}\left(-\mu \cdot M + N\right)$ where N is a local martingale orthogonal to M. We deduce from Lemma 2.5.12 that $\widetilde{M} = -\mu \cdot M + N$ is a BMO martingale. As M and N are orthogonal, $\mu \cdot M \in BMO$. □

Remark 2.6.7. *Recall that for U of power type with relative risk aversion $1 - p$ with $p \in (-\infty, 0) \cup (0,1)$, we have $a = b = 1 - p \in (0,1) \cup (1, +\infty)$. The necessity of the BMO property for the boundedness of $L(p)$ was first observed in [Nut12c, Corollary 5.12]. Lemma 2.6.6 is thus an extension of this result to the class of utility functions satisfying $(G_{a,b,C})$ with $a > 1$ or $b < 1$.*

The BMO property of $\mu \cdot M$ turns out be sufficient for the boundedness of L if $a \geq 1$. However this is not the case for $a < 1$. Indeed, for $U(z) = \frac{z^p}{p}, z > 0, p \in (0,1)$ in which case $a = 1-p \in (0,1)$ and $L = L(p)$, examples illustrating the insufficiency of the BMO property of $\mu \cdot M$ for the boundedness of $L(p)$ have been constructed by [FMW12]. Before stating a sufficient sharp condition for the boundedness of $L(p)$ based on μ obtained in [FMW12], let us provide for completeness some bounds of $L(p)$ that will also be needed in Chapter 3 for the study of the moments of L.

Lemma 2.6.8. *Let κ be the function defined as follows:*

$$\kappa : (0,+\infty) \ni z \mapsto z^2 + \frac{1}{2}z + z\sqrt{z(z+1)} = \frac{1}{2}\left(z + \sqrt{z(z+1)}\right)^2. \tag{2.41}$$

Then for $p \in (0,1)$, we have

$$L_\sigma(p) \leq \left(\mathbb{E}\left[\exp\left(\kappa(\frac{p}{1-p})\int_\sigma^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\middle|\mathcal{F}_\sigma\right]\right)^{1-\sqrt{p}}, \quad \sigma \in \mathcal{T}(\mathbb{F}). \tag{2.42}$$

Proof. The proof relies on the dual representation of $L(p)$ given by Proposition 2.3.4 and the fact that $Z^\mu \in \mathcal{Y}^*$. Note that for every $\sigma \in \mathcal{T}(\mathbb{F})$

$$\frac{Z^\mu}{Z^\mu_\sigma} = \exp\left(-\int_\sigma^T \mu_s dM_s - \frac{1}{2}\int_\sigma^T \mu_s^\top \langle M\rangle_s \mu_s\right).$$

We consider the case $p \in (0,1)$. Let $r = \frac{1}{p}$ and $\sigma \in \mathcal{T}(\mathbb{F})$. By Proposition 2.3.4

$$L_\sigma^{\frac{1}{1-p}}(p) = \operatorname*{ess\,inf}_{Y \in \mathcal{Y}^*} \mathbb{E}\left[\left(\frac{Y_T}{Y_\sigma}\right)^{\frac{p}{p-1}}\middle|\mathcal{F}_\sigma\right] \leq \mathbb{E}\left[\left(\frac{Z^\mu_T}{Z^\mu_\sigma}\right)^{\frac{p}{p-1}}\middle|\mathcal{F}_\sigma\right] = \mathbb{E}\left[\left(\frac{Z^\mu_\sigma}{Z^\mu_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right]. \tag{2.43}$$

Setting for $\rho > 1, \overline{\rho} = \frac{\rho}{\rho-1}$ and $\beta = \frac{1}{2}\frac{\rho}{r-1}\left(\frac{\rho}{r-1}+1\right)$. Using Hölder's inequality and the supermartingale property of $\mathcal{E}\left(\frac{\rho}{r-1}\mu \cdot M\right)$, we have

$$\begin{aligned}
\mathbb{E}\left[\left(\frac{Z^\mu_\sigma}{Z^\mu_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] &= \mathbb{E}\left[\left(\exp\left(\frac{\rho}{r-1}\int_\sigma^T \mu_s dM_s + \frac{1}{2}\frac{\rho}{r-1}\int_\sigma^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\right)^{\frac{1}{\rho}}\middle|\mathcal{F}_\sigma\right] \\
&= \mathbb{E}\left[\left(\mathcal{E}\left(\frac{\rho}{r-1}\mu \cdot M\right)_{\sigma,T}\right)^{\frac{1}{\rho}}\left(\exp\left(\beta\int_\sigma^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\right)^{\frac{1}{\rho}}\middle|\mathcal{F}_\sigma\right] \\
&\leq \left(\mathbb{E}\left[\mathcal{E}\left(\frac{\rho}{r-1}\mu \cdot M\right)_{\sigma,T}\middle|\mathcal{F}_\sigma\right]\right)^{\frac{1}{\rho}}\left(\mathbb{E}\left[\exp\left(\beta\frac{\bar{\rho}}{\rho}\int_\sigma^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\middle|\mathcal{F}_\sigma\right]\right)^{\frac{1}{\bar{\rho}}} \\
&\leq \left(\mathbb{E}\left[\exp\left(\beta\frac{\bar{\rho}}{\rho}\int_\sigma^T \mu_s d\langle M\rangle_s \mu_s\right)\middle|\mathcal{F}_\sigma\right]\right)^{\frac{1}{\bar{\rho}}}.
\end{aligned} \tag{2.44}$$

Observe that $\beta\frac{\overline{\rho}}{\rho} = \frac{\rho}{r-1}\left(\frac{\rho}{r-1}+1\right)$ is a function of ρ and the left hand side of (2.44) is minimal for $\rho = 1+\sqrt{r}$. In this case, $\beta\frac{\overline{\rho}}{\rho} = \frac{1}{2}\left(\frac{1}{1-\sqrt{r}}\right)^2 = \frac{1}{2}\frac{p}{(1-\sqrt{p})^2} = \kappa(\frac{p}{1-p})$. Now (2.43) and (2.44) yield (2.42). ☐

We will now introduce the notion of a *critical exponent* of a continuous local martingale to derive sharp conditions for the boundedness of $L(p)$,

Definition 2.6.9. *Let N be a continuous martingale with $N_0 = 0$. The critical exponent of N denoted by $b(N)$ is given by*

$$b(N) \quad = \quad \sup \left\{ l \geq 0 \mid \sup_{\sigma \in \mathcal{T}(\mathbb{F})} \left|\left| \mathbb{E}\left[\exp\Big(l \left(\langle N \rangle_T - \langle N \rangle_\sigma \right) \Big) | \mathcal{F}_\sigma \right] \right|\right|_\infty < \infty \right\}. \qquad (2.45)$$

Remark 2.6.10. *The notion of critical exponent was introduced in [Kaz94] to investigate the denseness of some subspaces of BMO. Using the John-Nirenberg inequality [Kaz94, Theorem 2.2], one sees that a continuous local martingale N is a BMO martingale if and only if $b(N) > 0$. It has been shown in [FMW12, Lemma 6.2] that the supremum in (2.45) is never attained.*

By the definition of the critical exponent, it follows from Lemma 2.6.8 that for $p \in (0,1)$, $L(p)$ is bounded if $b\,(\mu \cdot M)$ admits $\kappa(\frac{p}{1-p})$ as a strict lower bound. The following result from [FMW12] shows that in general the lower bound $\kappa(\frac{p}{1-p})$ for $b\,(\mu \cdot M)$ cannot be improved.

Theorem 2.6.11. *[FMW12, Theorem 6.5] Let $p \in (0,1)$ and κ defined by (2.41). We have the following:*

i) $L(p)$ is bounded away from 0 and ∞, if $b\,(\mu \cdot M) > \kappa(\frac{p}{1-p})$.

ii) Assume that $n = 1$, M is a one dimensional Brownian motion and $\mathbb{F}$ the completion of the filtration generated by M. Then:

- *for every $\delta < \kappa(\frac{p}{1-p})$, there exists a price process $S = S_0\mathcal{E}(R)$ with R given by (2.39) such that $\delta < b\,(\mu \cdot M) < \kappa(\frac{p}{1-p})$ and the corresponding opportunity process $L(p)$, is unbounded.*
- *there exists a price process $S = S_0\mathcal{E}(R)$ with R given by (2.39) such that $b\,(\mu \cdot M) = \kappa(\frac{p}{1-p})$ and the corresponding opportunity process $L(p)$, is bounded.*

Coming back to general utility functions, we have the following result regarding the boundedness of L away from 0 and ∞.

Proposition 2.6.12. *Suppose that U satisfies $(G_{a,b,C})$ with $a \leq b$. Let κ be defined by (2.41). Assume either $a \geq 1$ and $\mu \cdot M \in BMO$ or $a \in (0,1)$ and $b\,(\mu \cdot M) > \kappa\left(\frac{1-a}{a}\right)$. Then there exist two strictly positive constants $\underline{l}, \bar{l}$ depending only on a, b, C and $||\mu \cdot M||_{BMO}$ such that*

$$\underline{l} \leq L \leq \bar{l}.$$

Proof. Assume that $a \geq 1$ and $\mu \cdot M \in BMO$. By [Kaz94, Theorem 2.4], Z^μ satisfies (A_r) for some $r > 1$ depending only $||\mu \cdot M||_{BMO}$. We obtain the existence of $\underline{l}$ and $\bar{l}$ by applying Lemma 2.2.6 and Theorem 2.4.5.
Assume that $a \in (0,1)$ and $b(\mu \cdot M) > \kappa(\frac{1-a}{a})$. We set $p = 1 - a \in (0,1)$. Then $\frac{p}{1-p} = \frac{1-a}{a}$ and by Theorem 2.6.11, $L(1-a)$ is bounded or equivalently by Proposition 2.3.5 there exists $Y \in \mathcal{Y}^*$ which satisfies $(A_{\frac{1}{1-a}})$. The assertion follows from Theorem 2.4.9. □

Remark 2.6.13. *We recall that for $a > 1$, the necessity of the BMO property of $\mu \cdot M$ is justified by Lemma 2.6.6. In the specific case where $a = b \in (0,1)$, Proposition 2.5.5 entails that the boundedness of L away from 0 and ∞, is equivalent to that of $L(1-a)$. The bound $\kappa\left(\frac{1-a}{a}\right)$ is therefore sharp in the sense of Theorem 2.6.11.*

Remark 2.6.14. *Let $\mathbb{Q} \in \mathcal{M}^e(S)$ with density $Z^{\mathbb{Q}}$. Due to the decomposition (2.39) of R, $Z^{\mathbb{Q}}$ is of the form $Z^{\mathbb{Q}} = \mathcal{E}(M^{\mathbb{Q}})$ where $M^{\mathbb{Q}} = -\mu \cdot M + N^{\mathbb{Q}}$ with $N^{\mathbb{Q}}$ a local martingale orthogonal to M, see [AS92, Sch95]. If the filtration $\mathbb{F}$ is continuous, $b(M^{\mathbb{Q}})$ is well defined and by orthogonality of M and $N^{\mathbb{Q}}$ one has $b(\mu \cdot M) \geq b(M^{\mathbb{Q}})$. Thus Proposition 2.6.12 remains valid if one assumes that $b(M^{\mathbb{Q}}) > \kappa(\frac{1-a}{a})$.*

We close this section by showing that under the assumptions of Proposition 2.6.12 guaranteeing the boundedness of L, the running maximum of $\widehat{X}$ and $\widehat{Y}$ at the terminal date admit moments of order $k > 1$ and the optimal trading strategy ν is such that $\nu \cdot M$ is a BMO martingale. Moreover, k as well as the BMO norm $||\nu \cdot M||_{BMO}$ can be controlled solely by a, b, C and $||\mu \cdot M||_{BMO}$. The tractability of the constant k is at the core of the stability result of the optimal wealth and dual optimizer in the topology of uniform convergence established in Theorem 2.7.5.

Theorem 2.6.15. *Suppose that U satisfies $(G_{a,b,C})$ with $a \leq b$. Let $\mathbb{Q}^\mu$ be the probability measure defined by (2.40), ν the optimal trading strategy and κ the function defined by (2.41). We consider the assertions:*

i) $a \geq 1$ and $\mu \cdot M \in BMO$.

ii) $a \in (0,1)$ and $b(\mu \cdot M) > \kappa(\frac{1-a}{a})$.

Assume that i) or ii) hold. Then:

A1. There exists a strictly positive constant C_{BMO} depending only on a, b, C and $||\mu \cdot M||_{BMO}$ such that

$$||\nu \cdot M||_{BMO} + ||\nu \cdot R||_{BMO(\mathbb{Q}^\mu)} \leq C_{BMO}. \tag{2.46}$$

A2. There exists $\gamma > 1$ and $\eta > 0$ depending only on a, b, C and $||\mu \cdot M||_{BMO}$ such that

$$\mathbb{E}^{\mathbb{Q}^\mu}\left[\widehat{X}_T^{\gamma}\right] + \mathbb{E}\left[\widehat{Y}_T^{\gamma}\right] \leq (x^\gamma + \widehat{Y}_0^\gamma)\eta. \tag{2.47}$$

Moroever, $\widehat{X} \in \mathcal{S}^\gamma(\mathbb{Q}^\mu)$ and $\widehat{Y} \in \mathcal{S}^\gamma$.

Proof. Assume that i) or ii) hold. First, we fix some constants. By Proposition 2.6.12, there exist strictly positive constants $\underline{l}, \bar{l}$ depending only on a, b and C such that

$$\underline{l} \leq L \leq \bar{l}. \tag{2.48}$$

As $\mu \cdot M$ is a continuous BMO-martingale, by Proposition 2.8.2, there exists $k > 1$ depending only on $||\mu \cdot M||_{BMO}$ such that $Z^\mu = \mathcal{E}(-\mu \cdot M)$ satisfies (A_k). Let r, p and K_r be defined as follows:

$$r := \max\left\{\frac{1+a}{a}, 1 + bk\right\},\ p := \frac{b}{r-1} \text{ and } K_r := ||L||_\infty^{\frac{1}{r-1}}\left(1 + C^{\frac{1}{r-1}}||L(p)||_\infty\right).$$

A1. To prove (2.46), we begin by deriving the equation describing the dynamics of $\widehat{X}\widehat{Y}$. We recall that $\widehat{X} = x\mathcal{E}\left(\int_0^\cdot \nu(dM + \langle M\rangle\mu)\right)$ and $\widehat{Y} = \widehat{Y}_0\mathcal{E}\left(-\mu \cdot M + N\right)$ where N is local martingale orthogonal to M. An application of Itô's formula to $\widehat{X}\widehat{Y}$ gives for $t \in [0,T]$

$$d(\widehat{X}_t\widehat{Y}_t) \quad = \quad \widehat{X}_t d\widehat{Y}_t + \widehat{Y}_{t-}d\widehat{X}_t + d[\widehat{X},\widehat{Y}]_t = \widehat{X}_t\widehat{Y}_{t-}\left((-\mu_t + \nu_t)dM_t + dN_t\right) \tag{2.49}$$

Thus $\widehat{X}\widehat{Y} = x\widehat{Y}_0\mathcal{E}\left(\int_0^\cdot(\nu - \mu)dM + N\right)$. As L is bounded and Z^μ satisfies (A_k), Theorem 2.5.10 implies that $\widehat{X}\widehat{Y}$ satisfies (A_r). More precisely, with $\alpha = \frac{1}{r-1}$ and $\beta \in (0, \alpha/b)$, the estimate (2.36) holds for every $\sigma \in \mathcal{T}(\mathbb{F})$, i.e.

$$\mathbb{E}\left[\left(\frac{\widehat{X}_\sigma\widehat{Y}_\sigma}{\widehat{X}_T\widehat{Y}_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \leq K_r + \max\{C^\beta, C^{\frac{\alpha}{a}}\}||L||_\infty^{\beta+\alpha}\left(1 + ||L||_\infty^{\frac{\alpha}{a}-\beta}\right) =: \overline{K}_r. \tag{2.50}$$

Note that as $\widehat{X}$ has continuous paths, the bounds (2.48) imply that $\underline{l}/\bar{l} \leq \frac{\widehat{X}\widehat{Y}}{\widehat{X}_-\widehat{Y}_-} = \frac{L}{L_-} \leq \bar{l}/\underline{l}$. We deduce from Proposition 2.2.7 that $(\nu - \mu)\cdot M + N \in BMO$. Moreover, by Proposition 2.8.1, there exists K_{BMO} depending only on $\underline{l}, \bar{l}, r$ and $\overline{K}_r$ such that $||(\nu-\mu)\cdot M + N||^2_{BMO} \leq K_{BMO}$. Since M and N are orthogonal, we have $(\nu - \mu)\cdot M \in BMO$ and

$$||(\nu - \mu)\cdot M||^2_{BMO} \leq K_{BMO}. \tag{2.51}$$

As $(\nu - \mu)\cdot M$ and $\mu \cdot M$ are BMO martingales, $\nu \cdot M$ is a BMO martingale. Moreover, using the binomial inequalities $(x+y)^2 \leq 2x^2 + 2y^2$ for every $x, y \in \mathbb{R}$, it follows from (2.51) that

$$||\nu \cdot M||^2_{BMO} \quad = \quad ||(\nu-\mu)\cdot M + \mu \cdot M||^2_{BMO} \leq 2K_{BMO} + 2||\mu\cdot M||^2_{BMO}. \tag{2.52}$$

Recall that $\nu \cdot R = \nu \cdot M - \langle \nu \cdot M, -\mu \cdot M\rangle$. Since $Z^\mu = \mathcal{E}\left(-\mu \cdot M\right)$ is the density process of $\mathbb{Q}^\mu$, Theorem 3.6 in [Kaz94] entails that $\nu \cdot R \in BMO(\mathbb{Q}^\mu)$. Furthermore, by Theorem 2 in [CM14] there exists a constant $K^{'}_{BMO}$ depending only on $||\mu \cdot M||_{BMO}$ such that

$$||\nu \cdot R||_{BMO(\mathbb{Q}^\mu)} \quad \leq \quad K^{'}_{BMO}||\nu \cdot M||_{BMO}. \tag{2.53}$$

Let $C_{BMO} = (1 + K^{'}_{BMO})\sqrt{2K_{BMO} + 2||\mu\cdot M||^2_{BMO}}$. Combining (2.52) and (2.53), we obtain (2.46).

A2. We recall that $\widehat{X} = x\mathcal{E}\left(\nu \cdot R\right)$. As $||\nu\cdot R||_{BMO(\mathbb{Q}^\mu)} \leq C_{BMO}$ and R has continuous paths, by Corollary 2.2.8 there exist positive constants $\gamma_1 > 1$, and η_1 depending only on C_{BMO} such that

$$\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{Q}^\mu}\left[\widehat{X}_t^{\gamma_1}\right] \leq \eta_1 x^{\gamma_1}.$$

Note that $\widehat{X}^\mu$ is a $\mathbb{Q}^\mu$-uniformly integrable martingale. Hence applying Doob's maximal inequality, we obtain that $\widehat{X} \in \mathcal{S}^{\gamma_1}(\mathbb{Q}^\mu)$.

We now turn our attention to $\widehat{Y}$. By Theorem 2.5.3, $\widehat{Y}$ satisfies (A_r) with constant K_r, i.e. for $\sigma \in \mathcal{T}(\mathbb{F})$ we have

$$\mathbb{E}\left[\left(\frac{\widehat{Y}_\sigma}{\widehat{Y}_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \quad \leq \quad K_r.$$

Following Theorem 2.5.11, $\widehat{Y} = \widehat{Y}_0\mathcal{E}(\widetilde{M})$ where $\widetilde{M}$ is a BMO-martingale. Using the bounds (2.48) and the continuity of $\widehat{X}$, we have $\underline{l}/\bar{l} \leq \frac{\widehat{Y}}{\widehat{Y}_-} = \frac{L}{L_-} = 1 + \Delta\widetilde{M} \leq \bar{l}/\underline{l}$. We infer from Proposition 2.8.1 that $||\widetilde{M}||^2_{BMO} \leq \overline{K}_{BMO}$ where $\overline{K}_{BMO}$ is a positive constant depending only on $\underline{l}, \bar{l}, r$ and K_r. Applying Corollary 2.2.8, there exist positive constants $\gamma_2 > 1$, and η_2 depending only on $\underline{l}, \bar{l}, r$ and K_r such that

$$\sup_{t\in[0,T]} \mathbb{E}\left[\widehat{Y}_t^{\gamma_2}\right] \leq \eta_2 \widehat{Y}_0^{\gamma_2}.$$

The martingale property of $\widehat{Y}$ and Doob's maximal inequalities ensure that $\widehat{Y} \in \mathcal{S}^{\gamma_2}$. We obtain (2.47) by choosing $\gamma \in (1, \min\{\gamma_1, \gamma_2\})$, $\eta = \eta_1^{\frac{\gamma}{\gamma_1}} + \eta_2^{\frac{\gamma}{\gamma_2}}$ and using Hölder's inequalities. □

2.7 Stability of the utility maximization problem

We work in this section with the same setup as in Section 2.6.2, i.e. we assume that $S = S_0\mathcal{E}(R)$ is continuous and R is given by the canonical decomposition (2.39). Our aim is to exploit the integrability properties for the optimizers obtained in Theorem 2.6.15 to establish stability results for the utility maximization problem w.r.t. misspecification in the initial position and risk preference. To this end, let $(U_m)_{m\in\mathbb{N}}$ (resp. U) be a sequence of utility functions defined on $(0,+\infty)$ and $(x_m)_{m\in\mathbb{N}} \subseteq (0,+\infty)$ (resp. $x \in (0,+\infty)$) a sequence of initial capitals. We suppose that

Assumption 2.7.1. *$\lim_{m\to+\infty} x_m = x$ and $\lim_{m\to+\infty} U_m = U$ pointwise.*

The question regarding the convergence of the sequence of the optimal terminal wealth $(X_T^m)_{m\in\mathbb{N}}$ corresponding to the sequences $(U_m)_{m\in\mathbb{N}}$ and $(x_m)_{m\in\mathbb{N}}$ has been investigated in the literature [JN04, Lar09, KŽ11]. Though the convergence of the sequences $(U_m)_{m\in\mathbb{N}}$ and $(x_m)_{m\in\mathbb{N}}$ appears necessary for a positive answer, it is not sufficient. For a counterexample, we refer to [Lar09]. A complement for sufficiency is given by the condition (UI) introduced in [KŽ11].

Definition 2.7.2. *We say that a sequence $(U_m)_{m\in\mathbb{N}}$ of utility functions defined on $(0,+\infty)$ satisfies (UI) if there exists $\mathbb{Q} \in \mathcal{M}^e(S)$ such that*

$$\forall\ y > 0,\ \left(\frac{d\mathbb{Q}}{d\mathbb{P}} V_m^+\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right)_{m\in\mathbb{N}} \text{ is uniformly integrable}$$

where for each $m \in \mathbb{N}$, V_m is the convex conjugate of U_m and $V_m^+ = \max\{V_m, 0\}$.

Let us recall the main result in [KŽ11] on the stability of the utility maximization problem.

Theorem 2.7.3. *[KŽ11, Theorem 2.6] Let $(U_m)_{m\in\mathbb{N}}$ (resp. U) be a sequence of utility functions and $(x_m)_{m\in\mathbb{N}} \subseteq (0,+\infty)$ (resp. $x \in (0,+\infty)$) satisfying Assumption 2.7.1. For each $m \in \mathbb{N}$, let u_m and v_m (resp. u and v) be the value function of the primal and dual problems (2.4-2.5) associated to U_m (resp. U) and $y_m = u_m'(x_m)$ (resp. $y = u'(x)$). For each $m \in \mathbb{N}$, let $\widehat{X}^m$ and $\widehat{Y}^m$ (resp. $\widehat{X}$ and $\widehat{Y}$) be the optimal wealth process and dual optimizer to the corresponding primal and dual problems with utility function U_m (resp. U). Suppose that $(U_m)_{m\in\mathbb{N}}$ satisfies the condition (UI). We have the following convergence in probability*

$$\lim_{m\to+\infty} \widehat{X}_T^m = \widehat{X}_T \text{ and } \lim_{m\to+\infty} \widehat{Y}_T^m = \widehat{Y}_T.$$

Moreover, $\lim_{m\to+\infty} \widehat{Y}_0^m = \lim_{m\to+\infty} y_m = y$.

Our goal in the sequel is to build on Theorem 2.7.3 to establish a convergence result for the sequences $(\widehat{X}^m)_{m\in\mathbb{N}}$ and $(\widehat{Y}^m)_{m\in\mathbb{N}}$. The main difficulty is to identify the appropriate topology in which to investigate this convergence. To overcome this, we will assume that for some fixed $a, b, C > 0$ with $a \le b$, U_m satisfies the growth condition $(G_{a,b,C})$ for all $m \in \mathbb{N}$. As a result, we will be able to embed under the additional assumptions of Theorem 2.6.15, the sequences $(\widehat{X}^m)_{m\in\mathbb{N}}, (\widehat{Y}^m)_{m\in\mathbb{N}}$ respectively in the normed spaces $\mathcal{S}^k(\mathbb{Q}^\mu)$ and $\mathcal{S}^k$ for some $k > 1$. The latter embedding confers us with the topology of uniform convergence as a suitable candidate for our investigation. We will also be interested in the convergence w.r.t. the semimartingale topology (see Section 2.2.1).

Before stating the main result, we give the following lemma which constitutes a sufficient condition for the condition (UI) under the growth condition $(G_{a,b,C})$. It will be useful for the proof of Theorem 2.7.5 below.

Lemma 2.7.4. *Let $(U_m)_{m\in\mathbb{N}}$ (resp. U) be a sequence of utility functions with $(U_m)_{m\in\mathbb{N}}$ converging pointwise to U. Suppose that there exists $a, b, C > 0$ with $a \le b$ such that U_m satisfies the condition $(G_{a,b,C})$ for every $m \in \mathbb{N}$. Then U satisfies the condition $(G_{a,b,C})$. If either $a \ge 1$ or $a \in (0,1)$ and Z^μ satisfies $\left(A_{\frac{1}{1-a}}\right)$, then $(U_m)_{m\in\mathbb{N}}$ satisfies the condition (UI).*

Proof. For $m \in \mathbb{N}$, and $x_1, x_2 \in (0, +\infty)$ with $x_1 \leq x_2$, by definition of the condition $(G_{a,b,C})$ for U_m we have

$$\frac{1}{C}\left(\frac{x_2}{x_1}\right)^a \leq \frac{U_m^{'}(x_1)}{U_m^{'}(x_2)} \leq C\left(\frac{x_2}{x_1}\right)^{\mathrm{b}}. \tag{2.54}$$

As $(U_m)_{m\in\mathbb{N}}$ is a sequence of concave functions, the pointwise convergence of $(U_m)_{m\in\mathbb{N}}$ implies that $(U_m^{'})_{m\in\mathbb{N}}$ converges pointwise as welll (see [Roc70, Theorem 25.7]). One deduces from (2.54) that U satisfies the condition $(G_{a,b,C})$. We now proceed to show that $(U_m)_{m\in\mathbb{N}}$ satisfies the condition (UI). We will make use of the bounds for utility functions satisfying the condition $(G_{a,b,C})$ obtained in the of proof Lemma 2.3.8. Let $p \in (0, \frac{1}{2}), \overline{x} = (\frac{1}{p})^{\frac{1}{p}}$ and $x_0 = 1 + \overline{x}$. Since $(U_m)_{m\in\mathbb{N}}$ and $(U_m^{'})_{m\in\mathbb{N}}$ converge pointwise, we have

$$\overline{U} = \sup_{m\in\mathbb{N}} |U_m(x_0)| < +\infty \text{ and } \overline{U'} = \sup_{m\in\mathbb{N}} U_m^{'}(x_0) < +\infty.$$

We consider separately the cases $a > 1, a = 1$ and $a \in (0, 1)$.
Assume that $a > 1$. Let $m \in \mathbb{N}$ and $y > 0$. Then by (2.11), $U_m \leq U_m(x_0) + CU_m^{'}(x_0)\frac{x_0}{a-1}$. Hence

$$V_m(y) \quad = \quad \sup_{x>0}(U_m(x) - xy) \leq \overline{U} + C\overline{U'}\frac{x_0}{a-1}.$$

Let $\mathbb{Q} \in \mathcal{M}^e(S)$. Clearly, we have $\frac{d\mathbb{Q}}{d\mathbb{P}} V_m^+\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \leq \left(\overline{U} + C\overline{U'}\frac{x_0}{a-1}\right)\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^1$. As m is arbitrary, $\left(\frac{d\mathbb{Q}}{d\mathbb{P}} V_m^+\left(y\frac{d\mathbb{Q}}{d\mathbb{P}}\right)\right)_{m\in\mathbb{N}}$ is uniformly integrable. Thus $(U_m)_{m\in\mathbb{N}}$ satisfies (UI).

Assume that $a = 1$ and $\mu \cdot M \in BMO$. Let $m \in \mathbb{N}$ and $y > 0$. The condition $(G_{a,b,C})$ implies that U_m satisfies (2.13) which leads to the following upper bound for $V_m(y)$:

$$\begin{aligned} V_m(y) \quad &= \quad \sup_{x>0}(U_m(x) - xy) \leq U_m(x_0) + \sup_{x>0}(Cx_0U_m^{'}(x_0)x^p - xy) \\ &\leq \quad U_m(x_0) + (1-p)(1/p)^{\frac{p}{p-1}}(Cx_0U_m^{'}(x_0))^{\frac{1}{1-p}} y^{\frac{p}{p-1}} \\ &\leq \quad \overline{U} + (1-p)(1/p)^{\frac{p}{p-1}}(Cx_0\overline{U'})^{\frac{1}{1-p}} y^{\frac{p}{p-1}}. \end{aligned}$$

Using the above bound, we have $Z_T^\mu V_m^+(yZ_T^\mu) \leq Z_T^\mu \overline{U} + (1-p)(1/p)^{\frac{p}{p-1}}(Cx_0\overline{U'})^{\frac{1}{1-p}} y^{\frac{p}{p-1}}(Z_T^\mu)^{\frac{2p-1}{p-1}}$. The BMO property of $\mu \cdot M$ entails that $\mathbb{E}[Z_T^\mu] < +\infty$. As $p \in (0, \frac{1}{2})$, we have $r = \frac{2p-1}{p-1} \in (0, 1)$ and Hölder's inequality gives

$$\mathbb{E}[(Z_T^\mu)^r] \leq (\mathbb{E}[Z_T^\mu])^{\frac{1}{r}} < +\infty.$$

The family $(Z_T^\mu V_m^+(yZ_T^\mu))_{m\in\mathbb{N}}$ being $\mathbb{P}$-a.s. bounded by an L^1-integrable random variable, it is uniformly integrable. Since Z^μ is the density of the ELMM $\mathbb{Q}^\mu$ and $y > 0$ is arbitrary, $(U_m)_{m\in\mathbb{N}}$ satisfies (UI).
Assume that $a \in (0, 1)$ and Z^μ satisfies $\left(A_{\frac{1}{1-a}}\right)$. Let $m \in \mathbb{N}$ and $y > 0$. Due to the condition $(G_{a,b,C})$, (2.12) holds U_m for every $m \in \mathbb{N}$. Thus for $m \in \mathbb{N}$, and $y > 0$, we have

$$\begin{aligned} V_m(y) \quad &= \quad \sup_{x>0}(U_m(x) - xy) \leq U_m(x_0) + \sup_{x>0}\left(Cx_0U_m^{'}(x_0)\frac{x^{1-a}}{1-a} - xy\right) \\ &\leq \quad U_m(x_0) + \frac{a}{1-a}(Cx_0^aU^{'}(x_0))^{\frac{1}{a}} y^{\frac{a-1}{a}} \leq \overline{U} + \frac{a}{1-a}(Cx_0^a\overline{U'})^{\frac{1}{a}} y^{\frac{a-1}{a}}. \end{aligned}$$

The above inequality yields the following upper bound for $Z_T^\mu V_m^+(yZ_T^\mu)$, for $m \in \mathbb{N}$ and $y > 0$:

$$Z_T^\mu V_m^+(yZ_T^\mu) \leq \overline{U}Z_T^\mu + \frac{a}{1-a}\left(Cx_0^a\overline{U'}\right)^{\frac{1}{a}}(Z_T^\mu)^{\frac{2a-1}{a}}.$$

To conclude that $(Z_T^\mu V_m^+(yZ_T^\mu))_{m\in\mathbb{N}}$ is uniformly integrable, it suffices to show that $(Z_T^\mu)^{\frac{2a-1}{a}}$ is integrable. Now for $a\in[\frac{1}{2},1)$, we have $\frac{2a-1}{a}\in[0,1)$ and an application of Hölder's inequality gives

$$\mathbb{E}\left[(Z_T^\mu)^{\frac{2a-1}{a}}\right]\le(\mathbb{E}\left[Z_T^\mu\right])^{\frac{a}{2a-1}}<+\infty.$$

Assume that $a\in(0,\frac{1}{2})$. Clearly, $(Z_T^\mu)^{\frac{2a-1}{a}}=\left(\frac{1}{Z_T^\mu}\right)^{\frac{1-2a}{a}}=\left(\frac{1}{Z_T^\mu}\right)^{\frac{1}{\frac{1-a}{1-2a}-1}}$. Set $r=\frac{1-a}{1-2a}$. We have $r>\frac{1}{1-a}$. Since Z^μ satisfies $\left(A_{\frac{1}{1-a}}\right)$, it satisfies (A_r). Consequently,

$$\mathbb{E}\left[(Z_T^\mu)^{\frac{2a-1}{a}}\right]=\mathbb{E}\left[\left(\frac{1}{Z_T^\mu}\right)^{\frac{1}{r-1}}\right]<+\infty.$$

The proof is complete. □

The following theorem gives the convergence of the sequences $(\widehat{X}^m)_{m\in\mathbb{N}}$ and $(\widehat{Y}^m)_{m\in\mathbb{N}}$ in the semimartingale topology and the topology of uniform convergence. It also provides a statement for the convergence of the sequence of optimal trading strategies $(\nu^m)_{m\in\mathbb{N}}$.

Theorem 2.7.5. *Let $(U_m)_{m\in\mathbb{N}}$ (resp. U) be a sequence of utility functions defined on $(0,+\infty)$ and $(x_m)_{m\in\mathbb{N}}\in(0,+\infty)$ (resp. $x\in(0,+\infty)$). We keep the notation of Theorem 2.7.3. For each $m\in\mathbb{N}$, let ν^m, L^m (resp. ν, L) be the optimal strategy and opportunity process corresponding the primal problem with utility function U_m (resp. U). In addition to Assumption 2.7.1, we suppose that*

i) there exist $a,b,C>0$ with $a\le b$ such that for each $m\in\mathbb{N}, U_m$ satisfies $(G_{a,b,C})$,

ii) either $a\ge1$ and $\mu\cdot M\in BMO$ or $a\in(0,1)$ and $b(\mu\cdot M)>\kappa(\frac{1-a}{a})$.

Set $\left(\widehat{Y}^m-\widehat{Y}\right)^*=\sup_{t\in[0,T]}|\widehat{Y}_t^m-\widehat{Y}_t|$ and $\left(\widehat{X}^m-\widehat{X}\right)^*=\sup_{t\in[0,T]}|\widehat{X}_t^m-\widehat{X}_t|$. Let $\mathbb{Q}^\mu$ be the probability measure defined by (2.40). Then:

A1. There exists $k_1>1$ such that

$$\lim_{m\to+\infty}\mathbb{E}^{\mathbb{Q}^\mu}\left[|(\widehat{X}^m-\widehat{X})^*|^{k_1}\right]+\lim_{m\to+\infty}\mathbb{E}\left[|(\widehat{Y}^m-\widehat{Y})^*|^{k_1}\right]=0. \tag{2.55}$$

A2. $\lim_{m\to+\infty}\widehat{X}^m=\widehat{X}$ in $\mathcal{S}_0$ and $\lim_{m\to+\infty}\widehat{Y}^m=\widehat{Y}$ in $\mathcal{S}_0$.

A3. We have

$$\lim_{m\to+\infty}\mathbb{E}\left[\left(\int_0^T(\nu_s^m-\nu_s)^\top d\langle M\rangle_s(\nu_s^m-\nu_s)\right)^{\frac{1}{2}}\right]=0. \tag{2.56}$$

A4. For each $t\in[0,T]$, $\lim_{m\to+\infty}\mathbb{E}\left[|L_t^m-L_t|\right]=0$.

For the proof of A3. we will rely on the following convergence result from [Kar10].

Theorem 2.7.6. *[Kar10, Theorem 2.5] Let $(Z^n)_{n\in\mathbb{N}}$ be a sequence of strictly positive càdlàg local martingales. For $n\in\mathbb{N}$, let ϑ^n be the* stochastic logarithm *of Z^n, i.e. $\vartheta^n=\int_0^\cdot(1/Z_-^n)dZ^n$. Assume that for each $n\in\mathbb{N}$, $Z_0^n=1$ and for every $t\in[0,T]$, we have $\lim_{n\to+\infty}Z_t^n=1$ in probability. Then for every $t\in[0,T]$, $\lim_{n\to+\infty}[\vartheta^n,\vartheta^n]_t=0$ in probability.*

Proof of Theorem 2.7.5. A1. First we show that there exists $k_1 > 1$ such that $(|\widehat{X}^m_T - \widehat{X}_T|^{k_1})_{m\in\mathbb{N}}$ is $\mathbb{Q}^\mu$-uniformly integrable and $(|\widehat{Y}^m_T - \widehat{Y}_T|^{k_1})_{m\in\mathbb{N}}$ is $\mathbb{P}$-uniformly integrable. Since $(U_m)_{m\in\mathbb{N}}$ and U satisfy the hypotheses of Lemma 2.7.4, U satisfies also the condition $(G_{a,b,C})$. We infer from Theorem 2.6.15 that there exist $k > 1$, and η depending only on a, b, C and $||\mu \cdot M||_{BMO}$ such that for each $m \in \mathbb{N}$

$$\mathbb{E}^{\mathbb{Q}^\mu}\left[|\widehat{X}^m_T - \widehat{X}_T|^k\right] + \mathbb{E}\left[|\widehat{Y}^m_T - \widehat{Y}_T|^k\right] \le \left(x^k_m + (\widehat{Y}^m_0)^k + x^k + \widehat{Y}^k_0\right)\eta.$$

$(U_m)_{m\in\mathbb{N}}$ satisfies the condition (UI) by Lemma 2.7.4. By Theorem 2.7.3, $(\widehat{Y}^m_0)_{m\in\mathbb{N}}$ is bounded and thus the right hand term in the above inequality is uniformly bounded. We deduce that

$$\sup_{m\in\mathbb{N}} \mathbb{E}^{\mathbb{Q}^\mu}\left[|\widehat{X}^m_T - \widehat{X}_T|^k\right] + \sup_{m\in\mathbb{N}} \mathbb{E}\left[|\widehat{Y}^m_T - \widehat{Y}_T|^k\right] < +\infty. \tag{2.57}$$

Let $k_1 \in (1, k)$ and $r = \frac{k}{k_1} > 1$. As $(|\widehat{X}^m_T - \widehat{X}_T|^{k_1})_{m\in\mathbb{N}} \subseteq L^r(\mathbb{Q}^\mu)$ and (2.57) is satisfied, we infer from de la Vallée-Poussin Theorem that $(|\widehat{X}^m_T - \widehat{X}_T|^{k_1})_{m\in\mathbb{N}}$ is $\mathbb{Q}^\mu$-uniformly integrable. Similarly, $(|\widehat{Y}^m_T - \widehat{Y}_T|^{k_1})_{m\in\mathbb{N}}$ is $\mathbb{P}$-uniformly integrable.

We now show (2.55). Theorem 2.7.3 implies that $\lim_{m\to+\infty} |\widehat{X}^m_T - \widehat{X}_T|^{k_1} + |\widehat{Y}^m_T - \widehat{Y}_T|^{k_1} = 0$ in probability. The $\mathbb{Q}^\mu$-uniform integrability property of $(|\widehat{X}^m_T - \widehat{X}_T|^{k_1})_{m\in\mathbb{N}}$ and the $\mathbb{P}$-uniform integrability property of $(|\widehat{Y}^m_T - \widehat{Y}_T|^{k_1})_{m\in\mathbb{N}}$ give

$$\lim_{m\to+\infty} \mathbb{E}^{\mathbb{Q}^\mu}\left[|\widehat{X}^m_T - \widehat{X}_T|^{k_1}\right] + \lim_{m\to+\infty} \mathbb{E}\left[|\widehat{Y}^m_T - \widehat{Y}_T|^{k_1}\right] = 0. \tag{2.58}$$

For every $m \in \mathbb{N}, \widehat{X}^m - \widehat{X}$ is a $\mathbb{Q}^\mu$-uniformly integrable martingale and thus $|\widehat{X}^m - \widehat{X}|$ is a positive $\mathbb{Q}^\mu$ submartingale. Applying Doob's maximal inequality and using (2.58), one obtains

$$\lim_{m\to+\infty} \mathbb{E}^{\mathbb{Q}^\mu}\left[|(\widehat{X}^m - \widehat{X})^*|^{k_1}\right] \quad\le\quad \lim_{m\to+\infty}\left(\frac{k_1}{k_1 - 1}\right)^{k_1} \mathbb{E}^{\mathbb{Q}^\mu}\left[|\widehat{X}^m_T - \widehat{X}_T|^{k_1}\right] = 0.$$

For each $m \in \mathbb{N}$, $\widehat{Y}^m - \widehat{Y}$ is a $\mathbb{P}$-uniformly integrable martingale and thus $|\widehat{Y}^m - \widehat{Y}|$ is a $\mathbb{P}$-submartingale. Relying once more on Doob's maximal inequality and (2.58), it holds that $\lim_{m\to+\infty} \mathbb{E}\left[|(\widehat{Y}^m - \widehat{Y})^*|^{k_1}\right] = 0$. This completes the proof of (2.55).

A2. Recall that $(\widehat{X}^m - \widehat{X})_{m\in\mathbb{N}}$ is a sequence of $\mathbb{Q}^\mu$- martingales. By the Burkholder-Davis-Gundy (BDG) inequalities, there exist two positive constants $\underline{K}$ and $\overline{K}$ such that for every $m \in \mathbb{N}$, we have

$$\underline{K}\mathbb{E}^{\mathbb{Q}}\left[(\widehat{X}^m - \widehat{X})^*\right] \le \mathbb{E}^{\mathbb{Q}^\mu}\left[[\widehat{X}^m - \widehat{X}]^{\frac{1}{2}}_T\right] \le \overline{K}\mathbb{E}^{\mathbb{Q}}\left[(\widehat{X}^m - \widehat{X})^*\right].$$

The above inequalities together with (2.55) imply that

$$\lim_{m\to+\infty} \mathbb{E}^{\mathbb{Q}^\mu}\left[[\widehat{X}^m - \widehat{X}]^{\frac{1}{2}}_T\right] = 0. \tag{2.59}$$

The probability measures $\mathbb{P}$ and $\mathbb{Q}^\mu$ being equivalent, the semimartingale topology remains unchanged if one replaces $\mathbb{P}$ by $\mathbb{Q}^\mu$ (see [Eme79, Proposition 6]). We deduce from (2.59) and Proposition 2.2.9 that $(\widehat{X}^m)_{m\in\mathbb{N}}$ converges to $\widehat{X}$ in $\mathcal{S}_0$. The proof for $(\widehat{Y}^m)_{m\in\mathbb{N}}$ is based on similar arguments.

A3. We first show that

$$\lim_{m\to+\infty} \int_0^T (\nu^m_s - \nu_s)^\top d\langle M\rangle_s (\nu^m_s - \nu_s) = 0 \text{ in probability.} \tag{2.60}$$

We recall that $\widehat{X} = x\mathcal{E}(\nu \cdot R)$ and $\widehat{X} Z^\mu = x\mathcal{E}((\nu-\mu)\cdot M)$ is a uniformly integrable martingale as $(\nu-\mu)\cdot M \in BMO$. Let $\overline{\mathbb{Q}}$ be the probability measure equivalent to $\mathbb{P}$ and with density given by

$$\frac{d\overline{\mathbb{Q}}}{d\mathbb{P}} := \overline{Z} = \frac{\widehat{X} Z^\mu}{x}.$$

To show (2.60), we will apply Theorem 2.7.6 to an appropriate sequence of strictly positive local martingales. We begin by rewriting for each $m \in \mathbb{N}$, $\int_0^\cdot (\nu_s^m - \nu_s)^\top d\langle M\rangle_s (\nu_s^m - \nu_s)$ as the quadratic variation of the stochastic logarithm of a local martingale w.r.t. the measure $\overline{\mathbb{Q}}$. For $m \in \mathbb{N}$, set

$$\xi^m = \frac{x\widehat{X}^m}{x_m \widehat{X}} \text{ and } \zeta^m = \int_0^\cdot \frac{1}{\xi^m} d\xi^m.$$

Let $m \in \mathbb{N}$. Applying Itô's formula, we have for $t \in [0,T]$

$$\begin{aligned} d\left(\frac{1}{\widehat{X}_t}\right) &= -\frac{1}{\widehat{X}_t^2} d\widehat{X}_t + \frac{1}{\widehat{X}_t^3} d\langle \widehat{X}\rangle_t = -\frac{\nu_t}{\widehat{X}_t} dR_t + \frac{1}{\widehat{X}_t}(\nu_t)^\top d\langle M\rangle_t \nu_t, \\ d\xi_t^m = \frac{x}{x_m} d\left(\frac{\widehat{X}_t^m}{\widehat{X}_t}\right) &= \frac{x}{x_m}\widehat{X}_t^m d\left(\frac{1}{\widehat{X}_t}\right) + \frac{x}{x_m}\frac{1}{\widehat{X}_t} d\widehat{X}_t^m + \frac{x}{x_m} d\left\langle \frac{1}{\widehat{X}}, \widehat{X}^m \right\rangle_t \\ &= \xi_t^m \left((\nu_t^m - \nu_t) dM_t + (\nu_t^m - \nu_t)^\top d\langle M\rangle_t (\mu_t - \nu_t)\right). \end{aligned}$$

From the above equation and the definition of ζ^m, we have

$$\zeta^m = \int_0^\cdot (\nu_t^m - \nu_t) dM_t + (\nu_t^m - \nu_t)^\top d\langle M\rangle_t (\mu_t - \nu_t) \text{ and } [\zeta^m] = \int_0^\cdot (\nu_s^m - \nu_s)^\top d\langle M\rangle_s (\nu_s^m - \nu_s).$$

Clearly, ξ^m is a $\overline{\mathbb{Q}}$-local martingale since $\overline{Z}\xi^m = \frac{x}{x_m}\widehat{X}^m Z^\mu$ is $\mathbb{P}$-local martingale. Note that $\xi_0^m = 1$ and for every $t \in [0,T]$, we have $\xi_t^m > 0$. Moreover, by (2.55), as m goes to ∞, ξ_t^m tends to 1 in probability for every $t \in [0,T]$. We infer from Theorem 2.7.6 that as m goes to ∞, $[\zeta^m]_T$ tends to 0 in probability. In particular (2.60) holds.

Next we show that $\left([\zeta^m]_T^{\frac{1}{2}}\right)_{m\in\mathbb{N}}$ is uniformly integrable. By Theorem 2.6.15, there exists $C_{BMO} > 0$ such that $\sup_{m\in\mathbb{N}} ||\nu^m \cdot M||_{BMO} \le C_{BMO}$. Applying the binomial inequality $(x+y)^2 \le 2x^2 + 2y^2$ for $x, y \in \mathbb{R}$, we have for $m \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}\left[[\zeta^m]_T\right] &\le 2\mathbb{E}\left[\int_0^T (\nu_s^m)^\top d\langle M\rangle_s \nu_s^m\right] + 2\mathbb{E}\left[\int_0^T (\nu_s)^\top d\langle M\rangle_s \nu_s\right] \\ &\le 2||\nu^m \cdot M||_{BMO}^2 + 2||\nu \cdot M||_{BMO}^2 \le 2C_{BMO}^2 + 2||\nu \cdot M||_{BMO}^2 < +\infty. \end{aligned}$$

The above uniform bound implies that $\left([\zeta^m]_T^{\frac{1}{2}}\right)_{m\in\mathbb{N}}$ is uniformly integrable. As $\left([\zeta^m]_T^{\frac{1}{2}}\right)_{m\in\mathbb{N}}$ converges to 0 in probability, we also have L^1-convergence. Thus

$$\lim_{m\to+\infty} \mathbb{E}\left[\left(\int_0^T (\nu_s^m - \nu_s)^\top d\langle M\rangle_s (\nu_s^m - \nu_s)\right)^{\frac{1}{2}}\right] = \lim_{m\to+\infty} \mathbb{E}\left[[\zeta^m]_T^{\frac{1}{2}}\right] = 0.$$

A4. Since for each $m \in \mathbb{N}$, U_m satisfies $(G_{a,b,C})$, hypothesis ii), Proposition 2.6.12 implies that $(L^m)_{m\in\mathbb{N}}$ is uniformly bounded. Let $t \in [0,T]$. Recall that $L_t = \frac{\widehat{Y}_t}{U'(\widehat{X}_t)}$ and for each $m \in \mathbb{N}$, $L_t^m = \frac{\widehat{Y}_t^m}{U'(\widehat{X}_t^m)}$. As $(\widehat{Y}_t^m)_{m\in\mathbb{N}}$ and $(\widehat{X}_t^m)_{m\in\mathbb{N}}$ converge to 0 in probability by (2.55), we deduce that $\lim_{m\to+\infty} L_t^m = L_t$ in probability. The uniform boundedness of $(L^m)_{m\in\mathbb{N}}$ gives the result. $\square$

2.8 Appendix

This appendix collects some complements on BMO martingales and the Muckenhoupt's condition used in the proof of Theorem 2.6.15 and necessary for the proof of Corollary 2.2.8.

We begin with the following link between BMO martingales and Muckhenhoupt's condition (A_r) for some $r > 1$. The assertions can be found in [DDM79, Proposition 6] and [ISS79, Theorem 2].

Proposition 2.8.1. *Let M be a local martingale with $M_0 = 0$ and $Z = \mathcal{E}(M)$. The following assertions are true:*

i) Assume that there exist $r > 1$ and $K_r > 0$ such that for every $\sigma \in \mathcal{T}(\mathbb{F})$ we have

$$\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{1}{r-1}}\middle|\mathcal{F}_\sigma\right] \leq K_r. \tag{2.61}$$

Assume additionally that there exists $K > 1$ such that $\frac{1}{K} \leq \frac{Z}{Z_-} \leq K$. Then there exists a constant $K_{BMO} > 0$ depending only on r, K_r and K such that $||M||^2_{BMO} \leq K_{BMO}$.

ii) Suppose that there exists $\epsilon > 0$ such that $1 + \Delta M > \epsilon$ and M is a BMO-martingale. Then there exists $K^{'} > 0, r^{'}$ and $K_{r^{'}} > 0$ depending only on $||M||^2_{BMO}$ and $\epsilon > 0$ such that $1/K^{'} \leq Z/Z_- \leq K^{'}$. Moreover for every $\sigma \in \mathcal{T}(\mathbb{F})$, we have

$$\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{1}{r^{'}-1}}\middle|\mathcal{F}_\sigma\right] \leq K_{r^{'}}. \tag{2.62}$$

Proof. We begin with the first assertion following the same steps as in the proof of Proposition 6 in [DDM79]. We set $l = \frac{1}{r-1}$. Observe that $Z/Z_- = 1 + \Delta M$ and $1/K \leq Z/Z_- \leq K$ is equivalent to $1/K \leq 1 + \Delta M \leq K$. Thus $\Delta M \in [1/K - 1, K - 1]$. There exists $j \leq \frac{1}{2}$ such that

$$e^{jz^2} \quad \leq \quad \frac{e^z}{1+z},\ \forall\ z \in \mathcal{I} := [1/K - 1, K - 1]\,. \tag{2.63}$$

The process M is a BMO-martingale by Proposition 2.2.7. Let $\sigma \in \mathcal{T}(\mathbb{F})$. To obtain an estimate of $||M||^2_{BMO}$, we look at $\exp\left(\mathbb{E}\left[lj\left([M]_T - [M]_\sigma\right)|\mathcal{F}_\sigma\right]\right)$. Using the fact that M is a true martingale and Jensen's inequality, we have

$$\begin{aligned}\exp\left(\mathbb{E}\left[lj\left([M]_T - [M]_\sigma\right)|\mathcal{F}_\sigma\right]\right) &\leq \exp\left(\mathbb{E}\left[l\left(M_T - M_\sigma\right) + lj\left([M]_T - [M]_\sigma\right)|\mathcal{F}_\sigma\right]\right),\\ &\leq \mathbb{E}\left[\exp\left(l\left(M_T - M_\sigma\right) + lj\left([M]_T - [M]_\sigma\right)\right)|\mathcal{F}_\sigma\right].\end{aligned}$$

Now note that $[M]_T - [M]_\sigma = \langle M\rangle^c_T - \langle M\rangle^c_\sigma + \sum_{\sigma<u\leq T}|\Delta M_u|^2$. For $u \in [0,T]$, we set $\mathcal{J}_u := \left(\frac{e^{\Delta M_u}}{1+\Delta M_u}\right)^l$. As $\Delta M \in \mathcal{I}$ and $j \leq \frac{1}{2}$, we infer from the above estimate and (2.63) that

$$\begin{aligned}\exp\left(\mathbb{E}\left[lj\left([M]_T - [M]_\sigma\right)|\mathcal{F}_\sigma\right]\right) &\leq \mathbb{E}\left[\exp\left(l\left(M_T - M_\sigma\right) + \frac{l}{2}\left(\langle M\rangle^c_T - \langle M\rangle^c_\sigma\right) + lj\sum_{\sigma<u\leq T}\Delta M_u^2\right)\middle|\mathcal{F}_\sigma\right],\\ &\leq \mathbb{E}\left[\exp\left(l\left(M_T - M_\sigma\right) + \frac{l}{2}\left(\langle M\rangle^c_T - \langle M\rangle^c_\sigma\right)\right) \times \prod_{\sigma<u\leq T}\mathcal{J}_u\middle|\mathcal{F}_\sigma\right].\end{aligned}$$

The right hand term of the above inequality equals $\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^l\middle|\mathcal{F}_\sigma\right]$. Recalling that $l = \frac{1}{r-1}$, we deduce from hypothesis that $\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^l\middle|\mathcal{F}_\sigma\right] \leq K_r$. An application of the log function on both sides of the above estimates yields $\mathbb{E}\left[[M]_T - [M]_\sigma | \mathcal{F}_\sigma\right] \leq \frac{\log K_r}{lj}$. Since $|\Delta M| \leq K$ and $\Delta[M]_\sigma = |\Delta M_\sigma|^2 \leq K^2$, we have

$$\mathbb{E}\left[[M]_T - [M]_{\sigma-} | \mathcal{F}_\sigma\right] \leq \frac{\log K_r}{lj} + 4K^2 := K_{BMO}. \tag{2.64}$$

With the above choice of K_{BMO}, the first assertion is complete.
For the second assertion, we proceed as in [ISS79, Theorem 2]. As M is a BMO martingale, the jump process ΔM is uniformly bounded. W.l.o.g. we can assume that ϵ is sufficiently small and $1 + ||\Delta M||_\infty \leq \frac{1}{\epsilon}$. Choosing $K' = \frac{1}{\epsilon}$, we have $1/K' \leq Z/Z_- \leq K'$. Let $a > 0$ such that $\beta_a = (4a^2 + a)/\epsilon < 1/||M||^2_{BMO}$. Let $\sigma \in \mathcal{T}(\mathbb{F})$. Following the computations in the proof of Theorem 2 in [ISS79], we have

$$\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^a \middle| \mathcal{F}_\sigma\right] \leq \mathbb{E}\left[\exp\left(\beta_a\left([M]_T - [M]_{\sigma-}\right)\right) | \mathcal{F}_\sigma\right]$$

Applying John-Nirenberg's inequality, we obtain $\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^a \middle| \mathcal{F}_\sigma\right] \leq \frac{1}{1-\beta_a||M||^2_{BMO}}$.
We chose $r' = 1 + \frac{1}{a}$ and $K_{r'} = \frac{1}{1-\beta_a||M||^2_{BMO}}$. □

The following lemma gives an equivalent characterization of the Muckhenhoupt's condition.

Lemma 2.8.2. *Let Z be a strictly positive semimartingale, $r > 1$ and $C > 0$. The following are equivalent:*

a) Z satisfies the condition (A_r) with constant C, i.e. $\forall \sigma \in \mathcal{T}(\mathbb{F})$ we have $\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{1}{r-1}} \middle| \mathcal{F}_\sigma\right] \leq C$.

b) For every positive martingale Y and $\sigma \in \mathcal{T}(\mathbb{F})$, we have

$$Z_\sigma Y^r_\sigma \leq C\mathbb{E}\left[Z_T Y^r_T | \mathcal{F}_\sigma\right]. \tag{2.65}$$

Proof. See Proposition 1 and the remarks (more precisely inequality (9)) in [DDM79]. □

The following result is a weak version of Gehring's Lemma [DDM79, Proposition 4] and the main result to prove Corollary 2.2.8.

Proposition 2.8.3. *Let Z be a positive càdlàg process. Assume that there exist positive constants $r > 1$ and k_1, k_2, k_3 such that for every $\sigma \in \mathcal{T}(\mathbb{F})$, we have*

$$\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^{\frac{1}{r-1}} \middle| \mathcal{F}_\sigma\right] \leq k_1, \ \mathbb{E}\left[\frac{Z_T}{Z_\sigma} \middle| \mathcal{F}_\sigma\right] \leq k_2, \ \text{and} \ \frac{Z}{Z_-} \leq k_3. \tag{2.66}$$

Then there exist $\gamma > 1$, and K_b depending only on r, k_1, k_2 and k_3 such that

$$\left(\mathbb{E}\left[Z^\gamma_T\right]\right) \leq K_b Z^\gamma_0. \tag{2.67}$$

Proof. For the proof, we extend processes $\overline{Z}$ defined on $[0, T]$ to $[0, +\infty]$ by setting $\overline{Z}_t = \overline{Z}_T, t \in (T, +\infty]$. Similarly, we extend the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ to $(\mathcal{F}_t)_{0\leq t\leq+\infty}$ with $\mathcal{F}_\infty = \sigma\left(\cup_{t\geq0}\mathcal{F}_t\right)$. The sole purpose of this extension is to have a meaning for Z_τ on $\{\tau = +\infty\}$ where τ is the stopping time given by (2.68).
Let $\beta \in (0, 1)$ such that $(k_1\beta)^{\frac{1}{r}} \in (0, 1)$. We fix $1 - \delta = (k_1\beta)^{\frac{1}{r}}$. To prove the assertion, we will

apply Gehring's inequality, see [DDM79, Lemme 1]. To this end, let $\lambda > 0$, and τ the stopping time given by

$$\tau := \{t \geq 0 : Z_t > \lambda\}. \tag{2.68}$$

Let N be the martingale defined by $N_t = \mathbb{E}\left[1_{\{Z_T \leq \beta Z_\tau\}} | \mathcal{F}_t\right],\ t \in [0, +\infty]$. Note that

$$N_\infty = N_T = 1_{\{Z_T \leq \beta Z_\tau\}} \text{ and } N_\tau = \mathbb{E}\left[1_{\{Z_T \leq \beta Z_\tau\}} | \mathcal{F}_\tau\right].$$

As Z satisfies (A_r), we infer from Lemma 2.8.2 that

$$N_\tau^r Z_\tau \leq k_1 \mathbb{E}\left[Z_\infty N_\infty^r | \mathcal{F}_\tau\right] = k_1 \mathbb{E}\left[Z_T N_T^r | \mathcal{F}_\tau\right] = k_1 \mathbb{E}\left[Z_T 1_{\{Z_T \leq k_1 \beta Z_\tau\}} | \mathcal{F}_\tau\right] \leq k_1 \beta Z_\tau = (1-\delta)^r Z_\tau.$$

Consequently, we have $N_\tau = \mathbb{E}\left[1_{\{Z_T \leq \beta Z_\tau\}} | \mathcal{F}_\tau\right] \leq 1 - \delta$ which implies that

$$\mathbb{E}\left[1_{\{Z_T > \beta Z_\tau\}} 1_{\{\tau < +\infty\}} | \mathcal{F}_\tau\right] \geq a 1_{\{\tau < +\infty\}}. \tag{2.69}$$

Since $Z \leq k_3 Z_-$, we deduce that on the set $\{\tau < +\infty\} = \{\tau \leq T\}$, we have $Z_\tau \leq k_3 Z_{\tau-} \leq k_3 \lambda$. Using the inequality $\mathbb{E}\left[Z_T | \mathcal{F}_\tau\right] 1_{\{\tau \leq T\}} \leq k_2 Z_\tau 1_{\{\tau \leq T\}}$ and the previous one, together with (2.69), we obtain

$$\begin{aligned}
\mathbb{E}\left[Z_T 1_{\{Z_T > \lambda\}}\right] &\leq \mathbb{E}\left[Z_T 1_{\{\tau \leq T\}}\right] = \mathbb{E}\left[\mathbb{E}\left[Z_T 1_{\{\tau \leq T\}} | \mathcal{F}_\tau\right]\right] \\
&\leq k_2 \mathbb{E}^{\mathbb{Q}}\left[Z_\tau \mathbb{E}\left[1_{\{\tau \leq T\}} | \mathcal{F}_\tau\right]\right] \leq \lambda k_2 k_3 \mathbb{P}(\tau < +\infty) \leq \lambda \frac{k_2 k_3}{a} \mathbb{E}\left[1_{\{Z_T > \beta \lambda\}}\right].
\end{aligned}$$

Due to the last inequality, we infer from Gehring's inequality (see [DDM79, Lemme 1]) that there exists $\gamma > 1$ and K depending only on k_1, k_2, k_3 and β such that

$$\mathbb{E}\left[Z_T^\gamma\right] \leq K \left(\mathbb{E}\left[Z_T\right]\right)^\gamma.$$

As $\mathbb{E}\left[Z_T\right] \leq k_2 Z_0$, we obtain our desired inequality with $K_b = k_2^\gamma K$. □

We are now ready to prove Corollary 2.2.8 which we restate for convenience.

Corollary 2.8.4. *Let $K_{BMO} > 0$ and $\epsilon > 0$. There exists $p > 1$ and $S_p > 0$ depending only on K_{BMO} and ϵ such that for every local martingale M satisfying $||M||_{BMO}^2 \leq K_{BMO}$ and $\epsilon \leq 1 + \Delta M \leq \frac{1}{\epsilon}$, the process $Z = \mathcal{E}(M)$ satisfies*

$$\sup_{t \in [0,T]} \mathbb{E}\left[Z_t^p\right] \leq S_p.$$

Proof. Let M be a BMO-martingale satisfying $||M||_{BMO}^2 \leq K_{BMO}$ and $\epsilon \leq 1 + \Delta M \leq \frac{1}{\epsilon}$. Let $Z = \mathcal{E}(M)$. Then Z is a uniformly integrable martingale [ISS79, Theorem 2] and $Z/Z_- \leq \frac{1}{\epsilon}$. Let $a > 0$ such that $\beta_a = (4a^2 + a)/\epsilon \leq 1/||M||_{BMO}^2$. Following the proof of Proposition 2.8.1, for every $\sigma \in \mathcal{T}(\mathbb{F})$

$$\mathbb{E}\left[\left(\frac{Z_\sigma}{Z_T}\right)^a \Big| \mathcal{F}_\sigma\right] \leq \frac{1}{1 - \beta_a ||M||_{BMO}^2} \leq \frac{1}{1 - \beta_a K_{BMO}}.$$

The conditions of Proposition 2.8.3 are satisfied for Z with $r = 1 + \frac{1}{a}, k_1 = \frac{1}{1 - \beta_a K_{BMO}}, k_2 = 1$ and $k_3 = \frac{1}{\epsilon}$. We deduce that there exists $\gamma > 1$ and K_b depending only on r, k_1, k_2 and k_3 such that

$$\mathbb{E}\left[Z_T^\gamma\right] \leq K_b.$$

We now show that the last inequality holds at an arbitrary time $t \in [0,T]$. Let $t \in [0,T]$. We consider the stopped martingales $N := M_{\cdot\wedge t}$ and $P := \mathcal{E}(N) = Z_{\cdot\wedge t}$. As M is BMO martingale, N is a BMO-martingale and $||N||^2_{BMO} \leq ||M||^2_{BMO} \leq K_{BMO}$. Moreover, we also have $\epsilon \leq 1+\Delta N \leq \frac{1}{\epsilon}$. Since N satisfies the same conditions as M, P satisfies the same conditions as Z. Therefore, with the same constants γ and K_b as for Z, we have $\mathbb{E}\left[P_T^\gamma\right] = \mathbb{E}\left[Z_t^\gamma\right] \leq K_b$. The proof is complete. $\square$

3. FBSDEs systems related to utility maximization: analysis of solutions and risk aversion asymptotics

3.1 Introduction

In this chapter we consider the problem of maximization of utility from terminal wealth for a utility function U defined on the positive line, and asset prices modeled by a continuous semimartingale S. We provide a description of the optimal trading strategy by means of solutions to a system of forward-backward stochastic differential equations (FBSDEs), see (1.16). The forward equation describes the dynamical behavior of the optimal wealth $\widehat{X}$. The backward component of the system of FBSDEs describes the dynamics of the *generalized opporutunity process* L which corresponds to a reduced form of the *dual optimizer*, the solution to the associated dual problem. The system of FBSDEs is fully coupled except for logarithmic and power utilities, and the coefficients are non Lipschitz continuous. Using the *a priori estimates* of L derived in Chapter 2, we study normed spaces of solutions of the aforementioned system of FBSDEs under exponential moment and BMO conditions on the market price of risk, respectively. Under BMO conditions on the market price of risk, we prove the existence of a unique bounded solution.

Equipped with the knowledge of the normed spaces of solutions to our system of FBSDEs and the description of the optimal trading strategy, we exploit stability arguments from the theory of BSDEs to study the behavior of the optimal trading strategy as the coefficient of relative risk aversion uniformly approaches $c = 1$ or $c = +\infty$. For $c = 1$, we obtain that the limiting strategy is identical to the optimal trading strategy for logarithmic utility. In the case $c = +\infty$, we show that the limiting strategy corresponds to no trading at all. However, by rescaling the optimal trading strategy with the coefficient of relative risk aversion, one obtains in the limit the optimal trading strategy for the exponential utility maximization problem with unit absolute risk aversion. Our results generalize existing results valid for power utilities, i.e. $U(z) = \frac{z^p}{p}$, $p \in (-\infty, 0) \cup (0, 1)$, see [MT03b, Nut12c, MW13].

This chapter is structured as follows. In Section 3.2, we recall the utility maximization problem and some estimates for the generalized opportunity process L. In Section 3.3, we provide a dynamical description of the evolution of the pair $(\widehat{X}, L)$ by $(\widehat{X}, L)$ a system of forward backward stochastic differential equations (FBSDE). Tractable expressions of the optimal trading strategy and dual optimizer $\widehat{Y}$ are also given. Section 3.4 deals with the analysis of the integrability properties of solutions to systems of FBSDEs describing the joint dynamics of $(\widehat{X}, L)$. Several properties are deduced based on the market price of risk and the bounds on the derivatives of U. We study the risk aversion asymptotics of the optimizers in Section 3.5.

3.2 Preliminaries

We fix a time horizon $T \in (0, +\infty)$ and work with a filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$ with filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ satisfying the usual conditions of right-continuity and completeness, and $\mathcal{F}_0$ being trivial. $\mathcal{T}(\mathbb{F})$ denotes the set of stopping times valued in $[0, T]$. For a given probability measure $\mathbb{Q}$, $\mathbb{E}^{\mathbb{Q}}$ denotes the expectation w.r.t. the measure $\mathbb{Q}$. For $\mathbb{Q} = \mathbb{P}$, we will simply write $\mathbb{E}$. Let $n \in \mathbb{N}$. For $z \in \mathbb{R}^n$, $z^\top$ denotes its transpose and $||z|| = \left(z^\top z\right)^{\frac{1}{2}}$ its Euclidean norm. For an $\mathbb{R}^n$-valued semimartingale N and an $\mathbb{R}^n$-valued predictable integrand π, the stochastic integral, denoted by $\int_0^\cdot \pi dN$ or $\pi \cdot N$ is the scalar semimartingale with initial value zero given by $\int_0^\cdot \pi dN = \sum_{i=1}^m \int_0^\cdot \pi^i dN^i$. We denote by $\mathcal{L}(N)$ the set of $\mathbb{R}^n$-valued predictable integrands π, for

which $\int_0^{\cdot} \pi dN$ is well defined and by $\mathcal{S}$ the set of all real valued càdlàg semimartingales. For two real-valued càdlàg semimartingales N, R, $[N, R]$ denotes the quadratic covariation of N and R and $\langle N, R\rangle$ the predictable quadratic covariation of N and R when it exists[1]. We simply write $[N]$ for $[N, N]$ and $\langle N\rangle$ for $\langle N, N\rangle$. We recall that a real-valued local martingale R is orthogonal to an $\mathbb{R}^n$-valued local martingale $N = (N^1, \cdots, N^m)$ if and only if $[R, N^i]$ is a local martingale for $i = 1, \cdots, n$.
For $Z \in \mathcal{S}$, Z_- denotes the process of left limits, i.e. $Z_{t-} = \lim_{s\nearrow t} Z_s$ $(Z_{0-} = Z_0)$. and Z^* the running maximum process given by

$$Z_t^* = \sup_{s\le t} |Z_s|, \ t \in [0, T].$$

The Doleans-Dade exponential of a semimartingale Z will be denoted by $\mathcal{E}(Z)$.

3.2.1 The utility maximization problem

Let $R = (R^i)_{1\le i\le n}$ be an $\mathbb{R}^n$-valued continuous semimartingale with $R_0^i = 0$ for $i = 1, 2, \cdots, n$. We consider a financial market which consists of n assets and one bond. The bond is assumed to be constant and equal to 1. The joint price process for the n assets is modeled by a strictly positive $\mathbb{R}^n$-valued semimartingale $S = (S^i)_{1\le i\le n}$ with $S^i = S_0^i\mathcal{E}(R^i)$,$i = 1, 2\cdots n$. We denote by $\mathcal{M}^e(S)$ the set of equivalent local martingale measures for S. In order to exclude arbitrage opportunities in the sense of *No Free Lunch with Vanishing Risk* (see [DS94]), we assume throughout this whole chapter that

$$\mathcal{M}^e(S) \neq \emptyset. \tag{3.1}$$

The continuity of R and the condition $\mathcal{M}^e(S) \neq \emptyset$ entail that R satisfies the structure condition, i.e. there exists an $\mathbb{R}^n$-valued continuous local martingale M with $M_0 = 0$ and $\mu \in \mathcal{L}(M)$ such that

$$R \ = \ M + \int_0^{\cdot} d\langle M\rangle\mu. \tag{3.2}$$

From [JS03, II.2.9], there exists an $\mathbb{R}$-valued predictable increasing process K and a predictable process Σ with values in the set of all symmetric positive semidefinite $n \times n$ matrices such that for every $i, j \in \{1, 2, \cdots, n\}$, we have $\langle M\rangle^{i,j} = \int_0^{\cdot} \Sigma^{i,j} dK$. Moreover, $\Sigma = \sigma^\top\sigma$ where σ is a predictable process with values in the set of $n \times n$ matrices. We can w.l.o.g. take $K = \arctan\left(\sum_{i=1}^n \langle M^i, M^i\rangle\right)$ so that K is bounded[2]. Note that the factorization of $\langle M\rangle$ is not unique (see [JS03, II.2.9]) and our results do not depend on the specific choice of K. However, the boundedness of K is essential as we use BSDEs results from [MW12] for which this is required.
Let $\mathcal{P}$ be the predictable σ-field of $\mathbb{F}$-predictable sets on $[0, T] \times \Omega$. We consider the Doléans measure λ^K on $\mathcal{P}$, defined for $E \in \mathcal{P}$ by

$$\lambda^K(E) := \mathbb{E}\left[\int_0^T 1_E(t) dK_t\right].$$

Remark 3.2.1. *Denote by $\mathcal{N}$ the set of $\mathbb{R}^n$-valued predictable processes γ such that $\sigma\gamma = 0$, λ^K-a.e. It is clear that for $\pi, \nu \in \mathcal{L}(R)$ such that $\pi - \nu \in \mathcal{N}$, $\int_0^{\cdot} \pi dR$ and $\int_0^{\cdot} \nu dR$ are indistinguishable.*

[1] This is for example the case if N and R are locally square integrable martingales

[2] Indeed with $K = \arctan\left(\sum_{i=1}^n \langle M^i, M^i\rangle\right)$, for every $i, j \in \{1, 2, \cdots, n\}$, $\langle M^i, M^j\rangle$ is absolutely continuous w.r.t. K by Kunita-Watanabe inequality and the existence of Σ follows from Radon-Nikodym theorem.

We consider an investor with risk preference modeled by a utility function U defined on $(0, +\infty)$. By a utility function, we mean a strictly increasing, strictly concave and continuously differentiable function. Moreover, we will always assume that U satisfies Inada's conditions

$$\lim_{x \to +\infty} U^{'}(x) = 0 \text{ and } \lim_{x \to 0} U^{'}(x) = +\infty, \tag{3.3}$$

and U has reasonable asymptotic elasticity strictly less than 1, i.e.

$$AE[U] = \limsup_{x \to +\infty} \frac{xU^{'}(x)}{U(x)} < 1. \tag{3.4}$$

Let $x_0 > 0$ be the initial capital of the investor. A trading strategy is a predictable R-integrable $\mathbb{R}^n$-valued process $\pi = (\pi^i)_{1 \le i \le n}$, where π^i denotes the proportion of wealth invested in the asset $S^i, i = 1, \cdots, n$. The wealth process associated to an initial capital $x_0 > 0$ and the trading strategy π is defined by the equation

$$X_t^\pi \;=\; x_0 + \int_0^t \pi_u X_u^\pi dR_u, \;\; t \in [0, T].$$

The trading strategy π is said to be admissible[3] if $X^\pi \ge 0$ and we denote by $\mathcal{A}(x_0)$ the set of all admissible trading strategies with initial capital x_0. The aim of our investor is to trade so as to maximize his expected utility from terminal wealth. This corresponds to the stochastic control problem with value function u given by

$$u(x_0) \;=\; \sup_{\pi \in \mathcal{A}(x_0)} \mathbb{E}\left[U(X_T^\pi)\right], \;\; x_0 > 0. \tag{3.5}$$

An admissible trading strategy ν attaining the sup in (3.5) will be referred to as an optimal trading strategy and the corresponding wealth X^ν as the optimal wealth process.
Related to the problem (3.5) is a dual problem. Given $y > 0$, the dual domain $\mathcal{Y}(y)$ is given by

$$\mathcal{Y}(y) \;:=\; \{Y \ge 0 \mid Y_0 = y \text{ and } X^\pi Y \text{ is a supermartingale for every } \pi \in \mathcal{A}(1)\}\,.$$

Note that every $Y \in \mathcal{Y}(y)$ is a supermartingale since $0 \in \mathcal{A}(1)$. The value function v of the dual problem to (3.5) is

$$v(y) \;=\; \inf_{Y \in \mathcal{Y}(y)} \mathbb{E}\left[V\left(Y_T\right)\right], \;\; y > 0, \tag{3.6}$$

where for $y > 0$, $V(y) = \sup_{x>0}(U(x) - xy)$. We refer to $Y(y)$ attaining the inf in (3.6) as the dual optimizer. We have the following existence result for X^ν and $Y(y)$.

Theorem 3.2.2. *[KS99, Theorem 2.2.] Assume that $u(x) < +\infty$ for some $x > 0$. Then u and v are finite valued and continuously differentiable. Moreover, for $y = u^{'}(x_0)$, there exists $\nu \in \mathcal{A}(x_0)$ and $Y(y) \in \mathcal{Y}(y)$ such that*

$$u(x_0) = \mathbb{E}\left[U(X_T^\nu)\right] \;\; \text{and } v(y) = \mathbb{E}\left[V(Y_T(y))\right].$$

Furthermore, $Y_T(y) = U^{'}(X_T^\nu)$ and $X^\nu Y(y)$ is a uniformly integrable martingale.

[3] As R is continuous and $\mu \in \mathcal{L}(M)$, $\mathcal{L}(M) = \mathcal{L}(R)$ (see [CMS80, Théoreme 2]). Moreover, $\mathcal{E}(\int_0^\cdot \pi dR) \ge 0$ for every $\pi \in \mathcal{L}(R)$ and thus $\mathcal{A}(1) = \mathcal{L}(M)$

3.2.2 The generalized opportunity process

Suppose that $u(x_0) < +\infty$. Let $y = u^{'}(x_0)$, X^{ν} the optimal wealth process for (3.5) and $Y(y)$ the dual optimizer for (3.6). From now on, we set $\widehat{X} := X^{\nu}$ and $\widehat{Y} := Y(y)$. We recall the notion of generalized opportunity process introduced in Section 2.3.2 as the reduced form of the dual optimizer $\widehat{Y}$. More precisely, it is the process L given by

$$L := \frac{\widehat{Y}}{U'(\widehat{X})}. \tag{3.7}$$

Note that $L_T = 1$ and L depends only on $\widehat{X}$. Indeed, the equality $\widehat{Y}_T = U^{'}(\widehat{X}_T)$ and the martingale property of the product $\widehat{X}\widehat{Y}$ given by Theorem 3.2.2 imply that for $t \in [0, T]$

$$L_t = \frac{\widehat{Y}_t}{U'(\widehat{X}_t)} = \frac{\widehat{Y}_t \widehat{X}_t}{\widehat{X}_t U'(\widehat{X}_t)} = \frac{\mathbb{E}\left[\widehat{X}_T \widehat{Y}_T | \mathcal{F}_t\right]}{\widehat{X}_t U'(\widehat{X}_t)} \text{ and } L_T = \frac{\widehat{Y}_T}{U'(\widehat{X}_T)} = 1.$$

Let $p \in (-\infty, 0) \cup (0, 1)$ and $U(z) = \frac{z^p}{p}$, $z > 0$. Then by [Nut10, Propositions 3.1 and 3.4], L is the unique càdlàg semimartingale such that for every $\pi \in \mathcal{A}(x_0)$

$$\frac{1}{p}(X_t^{\pi}) L_t = \operatorname*{ess\,sup}_{\theta \in \mathcal{A}_t(x_0, \pi)} \mathbb{E}\left[\frac{1}{p}(X_T^{\theta}) | \mathcal{F}_t\right], \tag{3.8}$$

where for $t \in [0, T]$, $\mathcal{A}_t(x_0, \pi) = \{\theta \in \mathcal{A}(x_0) \mid \theta_s = \pi_s, s \in [0, t]\}$. We will use the notation $L(p)$ for L if $U(z) = \frac{z^p}{p}, z > 0$ and we will denote the corresponding value function by $\overline{u}_p$. For U of logarithmic type, i.e. $U(z) = \log z, z > 0$, we have $zU^{'}(z) = 1$ and thus $L = 1$. By convention, we will set

$$L(0) := 1.$$

Let us recall the growth condition $(G_{a,b,C})$ introduced in Section 2.3.2 for the purpose of deriving a priori bounds of L independently of the knowledge of $\widehat{X}$.

Definition 3.2.3. *Let $a, b, C > 0$ with $a \leq b$. We say that U satisfies the condition $(G_{a,b,C})$ if for every $x, y > 0$ with $x \leq y$, we have*

$$\frac{1}{C}\left(\frac{y}{x}\right)^a \leq \frac{U^{'}(x)}{U^{'}(y)} \leq C\left(\frac{y}{x}\right)^b.$$

Remark 3.2.4. *Let A_U be the relative risk aversion coefficient of U, i.e.*

$$A_U(x) = -\frac{xU^{''}(x)}{U^{'}(x)}, \ x > 0.$$

Note that if there exists $a, b \in (0, +\infty)$ with $a \leq A_U \leq b$, then U satisfies $(G_{a,b,1})$.

For convenience, we recall estimates of L given by Lemmas 2.4.4 and 2.4.6 from Chapter 2 for U satisfying the growth condition $(G_{a,b,1})$.

Lemma 3.2.5. *Suppose that U satisfies $(G_{a,b,1})$ for some a, b with $a \leq b$ and $u(x_0) < +\infty$.*

1. *Let $\alpha \in (0, a)$ and $\rho > 0$ such that $0 < \rho \leq \frac{\alpha}{b}$. Then for every $\tau \in \mathcal{T}(\mathbb{F})$, we have*

$$L_\tau(-\alpha) \leq L_\tau^{\rho} + L_\tau^{\frac{\alpha}{a}}.$$

2. *Assume that $a \in (0,1)$ and $\overline{u}_{1-a} < +\infty$. Let $\gamma \in (0, \min\{1, \frac{1}{b}\})$. Then for every $\tau \in \mathcal{T}(\mathbb{F})$, we have*

$$L_\tau \leq L_\tau^{1-\gamma} + L_\tau(1-a).$$

We will rely on Lemma 3.2.5 for the study of the normed properties of L in Section 3.4. We close this section with some theorems on regarding BMO martingales and the introduction of some normed spaces. The following theorem from [Kaz94] collects some useful properties of BMO martingales. For a probability measure $\mathbb{Q}$, $BMO(\mathbb{Q})$ denotes the space of BMO-martingales under the measure $\mathbb{Q}$.

Theorem 3.2.6. *[Kaz94, Theorems 3.1 and 3.6] Let N be a continuous $\mathbb{Q}$-local martingale with $N_0 = 0$ and $\Gamma = \mathcal{E}(N)$. The following assertions are true:*

a) *$N \in BMO(\mathbb{Q})$ implies that Γ is a $\mathbb{Q}$-uniformly integrable martingale and there exists $l > 1$ and C_2 depending only on $||N||_{BMO(\mathbb{Q})}$ such that for every $\tau \in \mathcal{T}(\mathbb{F})$,*

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\frac{\Gamma_T}{\Gamma_\tau}\right)^l \middle| \mathcal{G}_\tau\right] \leq C_2.$$

In other words, Γ satisfies the reverse Hölder's inequality (R_l) *w.r.t. the measure $\mathbb{Q}$.*

b) *Assume that Γ is a $\mathbb{Q}$-uniformly integrable martingale. The following are equivalent:*

   i) *Γ satisfies (R_k) for some $k \in (1, +\infty)$ w.r.t. the measure $\mathbb{Q}$.*

   ii) *$N \in BMO(\mathbb{Q})$.*

c) *Assume that $N \in BMO(\mathbb{Q})$ and let $\widehat{\mathbb{Q}}$ be the measure defined by $\left.\frac{d\widehat{\mathbb{Q}}}{d\mathbb{P}}\right|_{\mathcal{G}_T} = \Gamma_T$. Then for every $Z \in BMO(\mathbb{Q})$, $\widehat{Z} = Z - \langle Z, N\rangle \in BMO(\widehat{\mathbb{Q}})$. Moreover $||\widehat{Z}||_{BMO(\widehat{\mathbb{Q}})} \leq C_3 ||Z||_{BMO(\mathbb{Q})}$ where C_3 is a constant depending only on $||N||_{BMO(\mathbb{Q})}$.*

Let $r \geq 1$ and $\mathbb{Q}$ a probability measure equivalent to $\mathbb{P}$. We introduce the following spaces.

- $L^r(\mathbb{Q})$ (resp. $L^\infty(\mathbb{Q})$) is the space of $\mathcal{F}_T$-measurable real valued random variables H such that $||H||_{L^r(\mathbb{Q})} = \mathbb{E}^{\mathbb{Q}}[|H|^r] < +\infty$ (resp. $||H||_\infty = \operatorname{ess\,sup}_{\omega\in\Omega} |H(\omega)| < +\infty$).
- $\mathcal{S}^r(\mathbb{Q})$ (resp. $\mathcal{S}^\infty(\mathbb{Q})$) the space of real valued càdlàg semimartingales Z such that

$$||Z||_{\mathcal{S}^r(\mathbb{Q})} = \left(\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]} |Z_t|^r\right]\right)^{\frac{1}{r}} < +\infty \left(\text{resp. } ||Z||_\infty = \left|\left|\sup_{t\in[0,T]} |Z_t|\right|\right|_\infty < +\infty\right).$$

- $\Xi(\mathbb{Q})$ denotes set of càdlàg semimartingales Z such that for all $r \geq 1$

$$\mathbb{E}\left[\exp\left(r \sup_{t\in[0,T]} |Z_t|\right)\right] < +\infty.$$

- $\mathcal{H}^r(\mathbb{Q})$ denotes the set of $\mathbb{R}^n$-valued predictable processes Z such that

$$||Z||_{\mathcal{H}^r(\mathbb{Q})} := \left(\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T Z_s^\top d\langle M\rangle_s Z_s\right)^{\frac{r}{2}}\right]\right)^{\frac{1}{r}} < +\infty.$$

- $\mathcal{M}^r(\mathbb{Q})$ denotes the set of real valued local martingales R satisfying

$$||R||_{\mathcal{M}^r(\mathbb{Q})} := \left(\mathbb{E}^{\mathbb{Q}}\left[\langle R\rangle_T^{\frac{r}{2}}\right]\right)^{\frac{1}{r}}.$$

We will simply write $BMO, \mathbb{H}_1, L^r, L^\infty, \mathcal{S}^r, \mathcal{S}^\infty, \Xi$ and $\mathcal{M}^r$ if $\mathbb{P} = \mathbb{Q}$.

3.3 Tractable solution to the utility maximization problem

In this section, we derive equations describing the joint dynamics of the pair $(\widehat{X}, L)$ where $\widehat{X}$ is the optimal wealth process and L the generalized opportunity process defined by (3.7). The joint dynamics will enable us to obtain a tractable expression of the optimizers : optimal trading strategy ν, dual optimizer $\widehat{Y}$ and optimal wealth $\widehat{X}$. This leads to a suitable understanding of the links between the optimizers and their dependence w.r.t. input parameters such as the market price of risk μ, the risk preference U or its relative risk aversion coefficient A_U. Taking full advantage of this knowledge will allow us to study the continuity properties of the optimizers w.r.t. relative risk aversion coefficient in various topologies in Section 3.5. For power utilities, L is independent of $\widehat{X}$ and can be described as the backward component of the solution to a BSDE [HIM05, Mor09a, MT10, Nut12a]. Both the dynamics of $\widehat{X}$ and L result from the application of the martingale optimality principle. For general utilities, L depends on $\widehat{X}$ and a complete description of the dynamics of L relies on the joint dynamics of $(\widehat{X}, L)$. The tool of forward backward stochastic differential equations (hereafter FBSDEs) appears as a natural candidate to describe this dynamical behavior. We work under the following assumption on U:

Assumption 3.3.1. *U is three times continuously differentiable.*

Our system of FBSDEs of interest is the following:

$$\begin{cases} dX_t &= -\frac{U'(X_t)}{U''(X_t)}\left(\mu_t + \frac{Q_t}{P_{t_-}}\right) dM_t - \frac{U'(X_t)}{U''(X_t)}\left(\mu_t + \frac{Q_t}{P_{t_-}}\right)^\top d\langle M\rangle_t \mu_t, \ t \in [0,T], \\ X_0 &= x_0, \\ dP_t &= Q_t dM_t + dN_t + P_{t_-}\left(1 - \frac{1}{2}\frac{U^{(3)}(X_t)U'(X_t)}{|U''(X_t)|^2}\right)\left(\mu_t + \frac{Q_t}{P_{t_-}}\right)^\top d\langle M\rangle_t \left(\mu_t + \frac{Q_t}{P_{t_-}}\right), \\ P_T &= 1. \end{cases} \tag{3.9}$$

We now define what we mean by a solution of this system.

Definition 3.3.2. *A solution of (3.9) is a quadruple (X, P, Q, N) where*

1. *X is a strictly positive and continuous semimartingale,*
2. *P is a strictly positive càdlàg semimartingale with $P_- > 0$,*
3. *$Q \in \mathcal{L}(M)$ and N is a local martingale orthogonal to M and such that the system of equations (3.9) holds.*

The system (3.9) is fully coupled and the coupling is encoded by the map

$$\Phi_U : (0,\infty) \ni z \mapsto 1 - \frac{1}{2}\frac{U^{(3)}(z)U'(z)}{|U''(z)|^2}. \tag{3.10}$$

Clearly if Φ_U is constant i.e. $\Phi_U = c, c \in \mathbb{R}$, (3.9) decouples into a forward and backward component, where the latter is independent of the former. The solvability of the resulting backward and forward components can be addressed using the results from standard BSDE theory (see [Kob00, Mor09a, MW12]) and SDE theory (see [Pro04]). For the logarithmic utility $U(z) = \log z, z > 0$, we have $\Phi_{\log} = 0$ and a solution to (3.9) is given by $(x\mathcal{E}(\mu \cdot R), 1, 0, 0)$. For U of power type, i.e. $U(z) = \frac{z^p}{p}, z > 0$ for some $p \in (-\infty, 0) \cup (0, 1)$, we have $\Phi_U = \frac{p}{2(p-1)}$. The corresponding backward equation takes the form

$$dP_t = Q_t dM_t + dR_t + \frac{p}{2(p-1)} P_{t-}\left(\mu_t + \frac{Q_t}{P_{t-}}\right)^\top d\langle M\rangle_t \left(\mu_t + \frac{Q_t}{P_{t-}}\right), \ t \in [0,T], \ P_T = 1. \tag{3.11}$$

The BSDE (3.11) is the Bellman equation for power utility, see [Nut12a]. It admits infinitely many solutions. We refer to [FMW12, Theorem 3.6] for such constructions. Thus in general we expect (3.9) to admit infinitely many solutions in the sense of Definition 3.3.2. Therefore, we need to restrict to a particular class of solutions in order to address the question of uniqueness. We now introduce the following concept of solution that will lead to uniqueness.

Definition 3.3.3. *A solution (X, P, Q, N) of (3.9) admits the martingale property if $XU^{'}(X)P$ is a martingale.*
Let $A = (X^1, P^1, Q^1, N^1)$ and $B = (X^2, P^2, Q^2, N^2)$ be pairs of solutions to (3.9). We say that $A = B$ if and only if we have $X^1 = X^2$,$P^1 = P^2$, $Q^1 \cdot M = Q^2 \cdot M$ and $N^1 = N^2$. We will sometimes denote a solution (X^1, P^1, Q^1, R^1) by (X^1, P^1) and simply omit further mentioning of the additional solution components.

Our interest in solutions with the martingale property stems from the fact that we want to identify a solution (X, P) of (3.9) with $(\widehat{X}, L)$. As $\widehat{X}U^{'}(\widehat{X})L$ is a martingale, the martingale property is thus necessary for the above identification to hold. It is the minimal condition required to ensure a unique solution and can be seen as a minimality property for the backward component P of a solution, see Remark 3.3.6. In order to show that $(\widehat{X}, L)$ is the unique solution to (3.9) with the martingale property, we need the following lemma which shows that L is a special semimartingale.

Lemma 3.3.4. *Suppose that $u(x_0) < +\infty$ and Assumption 3.3.1 holds. The process L defined by (3.7) is a special semimartingale. Moreover, we have $L > 0$ and $L_- > 0$.*

Proof. Clearly $\mathbb{E}[\widehat{Y}_T] < +\infty$ as $\widehat{Y} \in \mathcal{Y}(y)$. Since $\widehat{Y}_T = U^{'}(\widehat{X}_T)$ by Theorem 3.2.2 and U satisfies Inada's conditions (3.3), we deduce that $\widehat{X}_T$ and $\widehat{Y}_T$ are strictly positive. $\widehat{X}\widehat{Y}$ is therefore a uniformly integrable martingale with strictly positive terminal value. The minimum principle for supermartingales entails that $\widehat{X}\widehat{Y} > 0$ and $\widehat{X}_-\widehat{Y}_- > 0$. Hence $L = \frac{\widehat{Y}}{U'(\widehat{X})}$ satisfies $L > 0$ and $L_- > 0$. By Assumption 3.3.1, $\frac{1}{U'}$ is twice continuously differentiable and thus $\frac{1}{U'(\widehat{X})}$ is a semimartingale. S being continuous, $\widehat{X}$ is continuous and $\frac{1}{U'(\widehat{X})}$ is locally bounded. As L is the product of a locally bounded process $\frac{1}{U'(\widehat{X})}$ and a special semimartingale $\widehat{Y}$, it is also a special semimartingale (see [EKQ95, Proposition 1.b]). □

As L is a special semimartingale, it admits a unique canonical decomposition $L = L_0 + M^L + A^L$, where M^L is a local martingale starting at 0 and A^L a predictable process of finite variation satisfying $A_0 = 0$. The process M being continuous, M^L admits a Kunita-Watanabe decomposition w.r.t. M (see [AS93]), i.e. there exists $Z^L \in \mathcal{L}(M)$ and N^L a local martingale orthogonal to M such that $M^L = \int_0^\cdot Z^L dM + N^L$. One can therefore rewrite the canonical decomposition of L as follows

$$L = L_0 + \int_0^\cdot Z^L dM + N^L + A^L. \tag{3.12}$$

The following theorem gives the explicit structure of A^L and shows that $(\widehat{X}, L)$ is the unique solution of (3.9) with the martingale property.

Theorem 3.3.5. *Suppose that $u(x_0) < +\infty$ and Assumption 3.3.1 holds. Let Z^L and N^L be given as in (3.12). The following assertions hold:*

i) The optimal trading strategy ν satisfies the following relation which is implicit since $\widehat{X}$ and L depend on ν:

$$\nu = -\frac{U^{'}(\widehat{X})}{\widehat{X}U^{''}(\widehat{X})}\left(\mu + \frac{Z^L}{L_-}\right), \quad \lambda^K\text{-}a.e. \tag{3.13}$$

ii) The quadruple $\left(\widehat{X}, L, Z^L, N^L\right)$ *is the unique solution of (3.9) with the martingale property.*

iii) The dual optimizer is given by

$$\widehat{Y} \;=\; U^{'}(x_0)L_0\mathcal{E}\left(\int_0^{\cdot} -\mu dM + \int_0^{\cdot} \frac{1}{L_-}dN^L\right), \tag{3.14}$$

and the martingale $X\widehat{Y}$ *has the form*

$$\widehat{X}\widehat{Y} \;=\; xU^{'}(x_0)L_0\mathcal{E}\left(\int_0^{\cdot}\left[-\mu + \frac{1}{A_U(\widehat{X})}\left(\mu + \frac{Z^L}{L_-}\right)\right]dM + \int_0^{\cdot}\frac{1}{L_-}dN^L\right), \tag{3.15}$$

where $A_U(z) = -\frac{zU^{''}(z)}{U^{'}(z)}, z \in (0, +\infty)$.

Proof. We recall that $\widehat{X} = X^{\nu} = x_0\mathcal{E}(\nu \cdot R)$ where $\nu \in \mathcal{A}(x_0)$ is the optimal trading strategy. In order to prove i) and ii), we first identify A^L and for this we use the fact that $\widehat{X}U^{'}(\widehat{X})L$ is a martingale. From Itô's formula, we have for $t \in [0, T]$

$$\begin{aligned}
dU^{'}(\widehat{X}_t) &= U^{''}(\widehat{X}_t)\widehat{X}_t\nu_t dM_t + \left(U^{''}(\widehat{X}_t)\widehat{X}_t\mu_t + \frac{1}{2}U^{(3)}(\widehat{X}_t)\widehat{X}_t^2\nu_t\right)^{\top} d\langle M\rangle_t\nu_t,\\
d\left(\widehat{X}_tU^{'}(\widehat{X}_t)\right) &= \left[U^{''}(\widehat{X}_t)\widehat{X}_t^2\nu_t + U^{'}(\widehat{X}_t)\widehat{X}_t\nu_t\right]dM_t + \left(U^{''}(\widehat{X}_t)\widehat{X}_t^2 + U^{'}(\widehat{X}_t)\widehat{X}_t\right)\nu_t^{\top}d\langle M\rangle_t\mu_t\\
&\quad + \left(\frac{1}{2}U^{(3)}(\widehat{X}_t)\widehat{X}_t^3 + U^{''}(\widehat{X}_t)\widehat{X}_t^2\right)\nu_t^{\top}d\langle M\rangle_t\nu_t.
\end{aligned}$$

Using the decomposition (3.12) of L and the above equation, one obtains for $t \in [0, T]$

$$\begin{aligned}
d\left(\widehat{X}_tU^{'}(\widehat{X}_t)L_t\right) &= \widehat{X}_tU^{'}(\widehat{X}_t)dL_t + L_{t-}d\left(\widehat{X}_tU^{'}(\widehat{X}_t)\right) + d\left[\widehat{X}U^{'}(\widehat{X}), L\right]_t \qquad (3.16)\\
&= \widehat{X}_tU^{'}(\widehat{X}_t)\left[Z_t^LdM_t + dN_t^L\right] + Y_{t-}\left[U^{''}(\widehat{X}_t)\widehat{X}_t^2\nu_t + U^{'}(\widehat{X}_t)\widehat{X}_t\nu_t\right]dM_t\\
&\quad + dB_t^L,
\end{aligned}$$

with

$$\begin{aligned}
B_t^L \;=\; & \int_0^t \widehat{X}_sU^{'}(\widehat{X}_s)dA_s^L + \int_0^t\left(U^{''}(\widehat{X}_s)\widehat{X}_s^2 + U^{'}(\widehat{X}_s)\widehat{X}_s\right)\nu_s^{\top}\langle M\rangle_s\left(L_{s-}\mu_s + Z_s^L\right)\\
& + L_{s_-}\left(\frac{1}{2}U^{(3)}(\widehat{X}_s)\widehat{X}_s^3 + U^{''}(\widehat{X}_s)\widehat{X}_s^2\right)\nu_s^{\top}d\langle M\rangle_s\nu_s. \qquad (3.17)
\end{aligned}$$

As L_-, Z^L, ν and $\widehat{X}$ are predictable, the finite variation process B^L is predictable. Since $\widehat{X}U^{'}(\widehat{X})L$ is a martingale, B^L is therefore a predictable local martingale of finite variation. Hence $B^L = 0$. Due to the equation (3.17) and the strict positivity of $\widehat{X}U^{'}(\widehat{X})$, we infer from [JS03, Theorem I.3.13] that for $t \in [0, T]$

$$\begin{aligned}
A_t^L \;=\; & -\int_0^t\left(1 + \frac{U^{''}(\widehat{X}_s)\widehat{X}_s}{U^{'}(\widehat{X}_s)}\right)\nu_s^{\top}d\langle M\rangle_s\left(L_{s-}\mu_s + Z_s^L\right) \qquad (3.18)\\
& - \int_0^t L_{s_-}\left(\frac{1}{2}\frac{U^{(3)}(\widehat{X}_s)\widehat{X}_s^2}{U^{'}(\widehat{X}_s)} + \frac{U^{''}(\widehat{X}_s)\widehat{X}_s}{U^{'}(\widehat{X}_s)}\right)\nu_s^{\top}d\langle M\rangle_s\nu_s.
\end{aligned}$$

i) We now show (3.13). To achieve this, we exploit the fact that $X^{\pi}U^{'}(\widehat{X})L$ is a supermartingale for every $\pi \in \mathcal{A}(x_0)$. Let $\pi \in \mathcal{A}(x_0)$. An application of Itô's formula yields for $t \in [0, T]$

$$\begin{aligned}
d\left(U^{'}(\widehat{X}_t)L_t\right) \;=\; & \left(U^{'}(\widehat{X}_t)Z_t^L + L_{t-}U^{''}(\widehat{X}_t)\widehat{X}_t\nu_t\right)dM_t + U^{'}(\widehat{X}_t)dN_t^L \qquad (3.19)\\
& - U^{'}(\widehat{X}_t)\left(L_{t-}\mu_t + Z_t^L + L_{t-}\frac{U^{''}(\widehat{X}_t)\widehat{X}_t}{U^{'}(\widehat{X}_t)}\nu_t\right)^{\top}d\langle M\rangle_t\nu_t,
\end{aligned}$$

$$\begin{aligned} d\left(X_t^\pi U^{'}(\widehat{X}_t)L_t\right) &= X_t^\pi d\left(U^{'}(\widehat{X}_t)L_t\right) + U^{'}(\widehat{X}_t)L_- dX_t^\pi + d\left[X^\pi, U^{'}(\widehat{X})L\right]_t \\ &= \left(X_t^\pi U^{'}(\widehat{X}_t)Z_t^L + L_{t-}U^{''}(\widehat{X}_t)X_t^\pi \widehat{X}_t\nu_t + U^{'}(\widehat{X}_t)L_{t-}X_t^\pi\pi_t\right)dM_t \\ &\quad + X_t^\pi U^{'}(\widehat{X}_t)dN_t^L \\ &\quad + U^{'}(\widehat{X}_t)X_t^\pi(\pi_t-\nu_t)^\top d\langle M\rangle_t\left(L_{t-}\mu_t + Z_t^L + L_{t-}\frac{U^{''}(\widehat{X}_t)\widehat{X}_t}{U^{'}(\widehat{X}_t)}\nu_t\right). \end{aligned}$$

The process $C^\pi = \int_0^\cdot X_t^\pi U^{'}(\widehat{X}_t)(\pi_t-\nu_t)^\top d\langle M\rangle_t\left(L_{t-}\mu_t + L_{t-}\frac{U^{''}(\widehat{X}_t)\widehat{X}_t}{U^{'}(\widehat{X}_t)}U^{'}(\widehat{X}_t)\nu_t\right)$ is predictable and of finite variation. We have $X^\pi U^{'}(\widehat{X})L = M^\pi + C^\pi$ where M^π is a local martingale. As $X^\pi U^{'}(\widehat{X})L$ is a supermartingale, it follows from Doob-Meyer's decomposition theorem that C^π is decreasing. Thus for every $t\in[0,T]$

$$\int_0^t X_s^\pi U^{'}(\widehat{X}_s)(\pi_s-\nu_s)^\top d\langle M\rangle_s\left(L_{s-}\mu_s + Z_s^L + L_{s-}\frac{U^{''}(\widehat{X}_s)\widehat{X}_s}{U^{'}(\widehat{X}_s)}\nu_s\right) \le 0,\ \mathbb{P}\text{-a.s.}. \tag{3.20}$$

Note that (3.20) holds for arbitrary $\pi\in\mathcal{A}(x_0)$. The process L being càdlàg, L_- is left-continuous and has right limits. Hence L_- is locally bounded (see [DM82b, Remark VII. 32]). As $\widehat{X}$ is continuous and $Z^L,\mu,\nu\in\mathcal{L}(M)$, we deduce that $\zeta = \nu + L_-\mu + Z^L + L_-\frac{U^{''}(\widehat{X})\widehat{X}}{U^{'}(\widehat{X})}\nu\in\mathcal{A}(x_0)$. Inserting $\pi=\zeta$ in (3.20) and using the factorization $d\langle M\rangle = \sigma^\top\sigma dK$ we obtain

$$\int_0^T X_t^\pi U^{'}(\widehat{X}_t)\left\|\sigma_t\left(\mu_t L_{t-} + Z_t^L + L_{t-}\frac{U^{''}(\widehat{X}_t)\widehat{X}_t}{U^{'}(\widehat{X}_t)}\nu_t\right)\right\|^2 dK_t \le 0.$$

Since $X^\pi U^{'}(\widehat{X})$ is strictly positive, we deduce that

$$\nu \quad = \quad -\frac{U^{'}(\widehat{X})}{U^{''}(\widehat{X})\widehat{X}}\left(\mu + \frac{Z^L}{L_-}\right) + \gamma,\ \lambda^K\text{-a.e.} \tag{3.21}$$

with $\gamma\in\mathcal{N}$ (Remark 3.2.1). As $\int_0^\cdot \gamma dR = 0$, we can w.l.o.g assume that $\gamma=0$ in (3.21). Thus (3.13) holds.

ii) First we show that $\left(\widehat{X}, L, Z^L, N^L\right)$ is a solution to (3.9). With ν defined by (3.13), $\widehat{X}$ satisfies

$$\begin{cases} d\widehat{X}_t &= -\frac{U^{'}(\widehat{X}_t)}{U^{''}(\widehat{X}_t)}\left(\mu_t + \frac{Z_t^L}{L_{t-}}\right)dM_t - \frac{U^{'}(\widehat{X}_t)}{U^{''}(\widehat{X}_t)}\left(\mu_t + \frac{Z_t^L}{L_{t-}}\right)^\top d\langle M\rangle_t\mu_t, t\in[0,T], \\ X_0 &= x_0. \end{cases}$$

We recall that $L = L_0 + M^L + A^L$ with $M^L = \int_0^\cdot Z^L dM + N^L$ and A^L defined by (3.18). Inserting (3.13) into A^L, we obtain that for $t\in[0,T]$

$$\begin{cases} dL_t &= Z_t^L dM_t + dN_t^L + L_{t-}\left(1 - \frac{1}{2}\frac{U^{(3)}(\widehat{X}_t)U^{'}(\widehat{X}_t)}{|U^{''}(\widehat{X}_t)|^2}\right)\left(\mu_t + \frac{Z_t^L}{L_{t-}}\right)^\top d\langle M\rangle_t\left(\mu_t + \frac{Z_t^L}{L_{t-}}\right), \\ L_T &= 1. \end{cases}$$

The quadruple $\left(\widehat{X}, L, Z^L, N^L\right)$ is therefore a solution of (3.9). Moreover, $\widehat{X}U^{'}(\widehat{X})L = \widehat{X}\widehat{Y}$ is a uniformly integrable martingale by Theorem 3.2.2. Hence $(\widehat{X}, L, Z^L, N^L)$ has the martingale property.

We now prove uniqueness. Let (X,P,Q,N) be a solution of (3.9) with the martingale property. Then X is the wealth process associated to the strategy $-\frac{U^{'}(X)}{U^{''}(X)X}\left(\mu + \frac{Q}{P_-}\right)$. Itô's product rule gives

$$d\left(U^{'}(X_t)P_t\right) \quad = \quad U^{'}(X_t)P_{t-}\left(-\mu_t dM_t + \frac{1}{P_{t-}}dN_t\right),\ t\in[0,T].$$

Let $\pi \in \mathcal{A}(x_0)$. Applying Itô's formula once more yields for $t \in [0,T]$

$$d\left(X_t^\pi U'(X_t)P_t\right) = \left[X_t^\pi U'(X_t)Q_t + P_{t_-}\left[-X_t^\pi U'(X_t)\left(\mu_t + \frac{Q_t}{P_{t_-}}\right) + U'(X_t)X^\pi \pi_t\right]\right] dM_t + X_t^\pi U'(X_t) dN_t.$$

One sees that $X^\pi U'(X)P$ is a local martingale. Since it is positive, it is a supermartingale and therefore the difference $(X^\pi - X)\,U'(X)P$ is supermartingale with initial value 0. From the strict concavity of U, we obtain that $\mathbb{E}\left[U(X_T^\pi) - U(X_T)\right] \leq \mathbb{E}\left[(X_T^\pi - X_T)\,U'(X_T)P_T\right] \leq 0$. The process X is therefore an optimal wealth process. By the uniqueness of the optimal wealth, we have $X = \widehat{X}$. We infer that $\widehat{X}U'(\widehat{X})P$ and $\widehat{X}U'(\widehat{X})L$ are both martigales with the same terminal value. Hence, they are indistinguishable. Thus $P = L$ as $\widehat{X}U'(\widehat{X})$ is strictly positive. Since L is a special semimartingale, its canonical decomposition is unique. Thus $L_0 + M^L = L_0 + \int_0^\cdot QdM + N$. By uniqueness of the Kunita-Watanabe decomposition, we deduce that $\int_0^\cdot Z^L dM = \int_0^\cdot QdM$ and $N^L = N$. Uniqueness is proven.
iii) By definition of L, we have $\widehat{Y} = U'(\widehat{X})L$ whose dynamical behavior is described by (3.19). We obtain (3.14) by using the expression of ν given by (3.13). Noting that $B^L = 0$, we recall from (3.16) that for $t \in [0,T]$

$$d\left(\widehat{X}_t U'(\widehat{X}_t)L_t\right) = \widehat{X}_t U'(\widehat{X}_t)\left[Z_t^L dM_t + dN_t^L\right] + Y_{t_-}\left[U''(\widehat{X}_t)\widehat{X}_t^2 \nu_t + U'(\widehat{X}_t)\widehat{X}_t \nu_t\right] dM_t.$$

Inserting the expression of ν given by (3.13) in the above equation, we obtain the formula of $\widehat{X}\widehat{Y} = \widehat{X}U'(\widehat{X})L$ given by (3.15). The proof is complete. □

Remark 3.3.6. *i) Theorem 3.3.5 gives a precise description of the optimal portfolio in terms of the solution (3.9). A similar result in the setting of a continuous filtration appears in [HHI+14, ST14]. The derivation of the system of FBSDEs (3.9) relies on the variational calculus of Peng [Pen93] and requires additional integrability conditions on $\widehat{X}$ and bounds on the derivative of U. However, they work with a terminal random endowment which we do not consider here.*

ii) The relation (3.14) shows that $\widehat{Y}$ is a local martingale. This is a known fact for S with continuous paths, see [KLSX91, Lv07, KW16]. However, L allows for a concise description of $\widehat{Y}$. The dependence of $\widehat{Y}$ on the local martingale $\int_0^\cdot \frac{1}{L_-}dN^L$ highlights the necessity of a suitable knowledge of L to investigate the uniformly integrability of $\widehat{Y}$ as illustrated in Section 2.5.1.

iii) The martingale property attributes to L a minimality property. Indeed, let $L^+(\widehat{X})$ be the set of strictly positive processes P such that $P_T = 1$ and $\widehat{X}U'(\widehat{X})P$ is a local martingale. As $\widehat{X}U'(\widehat{X})L$ is a martingale, we deduce that $L \leq P$ for all $P \in L^+(X)$. Clearly L is the minimal element of $L^+(\widehat{X})$. The above minimality property has been observed in [Nut12a] for power utilities. There the set $L^+(\widehat{X})$ can be identified with the class of solutions to the Bellman BSDE (3.11).

iv) If the filtration $\mathbb{F}$ is continuous then L is a continuous semimartingale. If $\mathbb{F}$ is discontinuous, L might be discontinuous and its jumps are embedded into N^L.

Remark 3.3.7. *Theorem 3.3.5 above establishes an equivalence between the existence of a solution to (3.9) with the martingale property and the existence of a solution to the problem (3.5). In general, fully coupled systems of FBSDE admit solutions only for T sufficiently small. This occurs for (3.9) if $u < +\infty$ only for $T < T_\infty$ for some $T_\infty \in (0,\infty)$. See for example [KMK10, Section 2] or [KK04, Section 3.4] where S is given by the Heston model [Hes93] and U is the power utility.*

3.4 Normed spaces of the solution $(\widehat{X}, L, Z^L, N^L)$

Our aim in this section is to give the integrability properties of $(\widehat{X}, L, Z^L, N^L)$ based on conditions on the market price of risk μ and the functions A_U, Φ_U by identifying a suitable normed space in which $(\widehat{X}, L, Z^L, N^L)$ belongs to, e.g. $(\widehat{X}, L, Z^L, N^L) \in \mathcal{S}^r \times \mathcal{S}^r \times \mathcal{H}^r \times \mathcal{M}^r$ for some $r \geq 1$. The main difficulty in providing the integrability properties lies in the fact that $\widehat{X}$ is coupled to L. This coupling hinders the direct application of results from SDE theory [Pro04] and or BSDEs theory [Kob00, Mor09a, MW12] to embed $(\widehat{X}, L, Z^L, N^L)$ in a normed space. We will circumvent this coupling by exploiting partial estimates of L given by Lemma 3.2.5 to establish its integrability properties. We will then rely on the canonical decomposition of L and results from semimartingale theory [LLP80, MW12] to derive the integrability properties of $\widehat{X}, Z^L$ and N^L from those of L.

We will work under the following assumptions on U:

Assumption 3.4.1. *There exists $\underline{c} \in (0, +\infty)$ such that for every $x > 0$, $A_U(x) = -\frac{xU''(x)}{U'(x)}$ satisfies*

$$\underline{c} \leq A_U(x).$$

Assumption 3.4.2. *There exist $\underline{\alpha}, \overline{\alpha} \in (-\infty, \frac{1}{2})$ such that for every $x > 0$, $\Phi_U(x) = 1 - \frac{1}{2}\frac{U^{(3)}(x)U'(x)}{|U''(x)|^2}$ satisfies*

$$\underline{\alpha} \leq \Phi_U(x) \leq \overline{\alpha}.$$

Remark 3.4.3. *Under Assumption 3.4.1, the conditions $\underline{\alpha}, \overline{\alpha} \in (-\infty, \frac{1}{2})$ in Assumption 3.4.2 are not too restrictive. To see this, set $\frac{U'(0)}{U''(0)} := \lim_{x\downarrow 0} \frac{U'(x)}{U''(x)}$. Note that $A_U \geq \underline{c}$ implies that $0 \leq -\frac{U'(x)}{U''(x)} \leq \frac{x}{\underline{c}}$, $x > 0$. Taking the limit as $x \downarrow 0$ yields $\frac{U'(0)}{U''(0)} = 0$. Now $\Phi_U = \frac{1}{2} + \frac{1}{2}(\frac{U'}{U''})'$. Therefore $\underline{\alpha} \leq \Phi_U \leq \overline{\alpha}$ is equivalent to*

$$2\underline{\alpha} - 1 \leq \left(\frac{U'}{U''}\right)' \leq 2\overline{\alpha} - 1.$$

Using $\frac{U'(0)}{U''(0)} = 0$ and integrating $(\frac{U'}{U''})'$ between 0 and $x > 0$, one obtains

$$1 - 2\overline{\alpha} \leq 1/A_U(x) = -\frac{U'(x)}{xU''(x)} \leq 1 - 2\underline{\alpha}. \tag{3.22}$$

As $A_U > 0$, it follows that $\underline{\alpha} < \frac{1}{2}$. It is by now commonly acknowledged from the works of [Arr65] that the coefficient of absolute risk aversion $-\frac{U''}{U'}$ is decreasing. Assuming the latter property, one has $\left(\frac{U'}{U''}\right)' < 0$ and thus $\Phi_U = \frac{1}{2} + \frac{1}{2}\left(\frac{U'}{U''}\right)' < \frac{1}{2}$. Hence $\overline{\alpha} < \frac{1}{2}$. The restrictive condition in Assumption 3.4.2 is the boundedness from below of the function Φ_U.

Remark 3.4.4. *Under Assumptions 3.4.1 and 3.4.2, one deduces from (3.22) that*

$$a = \frac{1}{1 - 2\underline{\alpha}} \leq A_U \leq b = \frac{1}{1 - 2\overline{\alpha}}. \tag{3.23}$$

It follows from Remark 3.2.4 that U satisfies therefore $(G_{a,b,1})$ (see Definition 3.2.3). Note that $a \geq 1$ is equivalent to $\Phi_U \geq 0$ and $b \leq 1$ is equivalent to $\Phi_U \leq 0$.

The primary necessity of Assumptions 3.4.1 and 3.4.2 is to enhance U with the growth condition $(G_{a,b,1})$. As a result, we can apply Lemma 3.2.5 to obtain the integrability properties of L on the basis of those of $L(p), p \in (-\infty, 0) \cup (0, 1)$ as shown in Theorem 3.4.10 below. The secondary necessity of Assumptions 3.4.1 and 3.4.2 is to attribute convexity, locally Lipschitz and quadratic growth properties to the driver of the BSDE describing the dynamics of $\log L$, see Remark 3.4.5. These properties are necessary to infer the integrability properties of Z^L and N^L from those of L, see Propositions 3.4.12 and 3.4.14.

We recall that $\langle M\rangle = \sigma^\top\sigma$. By Theorem 3.3.5, the equation describing the dynamics of L is given by

$$dL_t \;=\; Z^L dM_t + dN_t^L + L_{t-}\Phi_U(\widehat{X}_t)\left(\mu_t + \frac{Z_t^L}{L_{t-}}\right)^\top d\langle M\rangle \left(\mu_t + \frac{Z_t^L}{L_{t-}}\right), \; L_T = 1. \quad (3.24)$$

Let F_U be defined for $(t, \omega, x, z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^n$ by

$$F_U(t, \omega, x, z) \;=\; -\Phi_U(x)||\sigma_t(\omega)(\mu_t(\omega) + z)||^2 + \frac{1}{2}||\sigma_t(\omega)z||^2, \quad (3.25)$$

and (ξ^L, Π^L) the pair defined by the change of variable

$$\xi^L = \frac{Z^L}{L-} \text{ and } \Pi^L = \int_0^\cdot \frac{1}{L-} dN^L. \quad (3.26)$$

Assuming L is continuous, an application of Itô's formula to $\log L$ using the equation above yields

$$d\log L_t \;=\; \xi_t^L dM_t + d\Pi_t^L - F_U(t, \widehat{X}_t, \xi_t^L)dK_t - \frac{1}{2}d\left[\Pi^L\right]_t, \; t \in [0, T], \; \log L_T = 0. \quad (3.27)$$

Remark 3.4.5. *Assumption 3.4.2 confers to F_U the following properties:*

1. *For all $(t, x) \in [0, T] \times (0, +\infty)$, the map $\mathbb{R}^n \ni z \mapsto F_U(t, x, z)$ is convex. As $\frac{1}{2} - \Phi_U > 0$, the convexity of $F_U(t, x, \cdot)$ follows from the decomposition*

$$F_U(t, x, z) \;=\; \left(-\Phi_U(x) + \frac{1}{2}\right)||\sigma_t z||^2 - \Phi_U(x)||\sigma_t\mu_t||^2 - 2\Phi_U(x)(\sigma_t\mu_t)^\top(\sigma_t z).$$

2. *F_U has quadratic growth in z. Indeed, let $\beta = \max\{|\underline{\alpha}|, |\overline{\alpha}|\}$ and $\delta > 0$. Young's inequality $2|z_1^\top z_2| \le \delta||z_1||^2 + \frac{1}{\delta}||z_2||^2, z_1, z_2 \in \mathbb{R}^n$, yields for $(t, x, z) \in [0, T] \times (0, +\infty) \times \mathbb{R}^n$,*

$$|F_U(t, x, z)| \le \beta\left(1 + \frac{1}{\delta}\right)||\sigma_t\mu_t||^2 + \frac{1}{2}(2\beta + 2\beta\delta + 1)||\sigma_t z||^2. \quad (3.28)$$

3. *F_U is as well locally Lipschitz in z since for $(x, z_1, z_2) \in (0, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n$ we have*

$$|F_U(t, x, z_1) - F_U(t, x, z_2)| \;\le\; \left(2\beta||\sigma_t\mu_t|| + \frac{2\beta+1}{2}||\sigma_t z_1|| + \frac{2\beta+1}{2}||\sigma_t z_2||\right)||\sigma_t\Delta z||,$$

with $\Delta z = z_1 - z_2$.

Let us gives some examples of utility functions satisfying Assumptions 3.4.1 and 3.4.2.

Example 3.4.6. *Let $p, q \in (-\infty, 0) \cup (0, 1)$ with $p \le q$.*

i) Let $U(z) = \log z + \frac{z^p}{p}, z > 0$*. Then for* $z > 0$*, we have*

$$\begin{aligned} A_U(z) &= \frac{1+(1-p)z^p}{1+z^p}, \quad \left(\frac{U'(z)}{U''(z)}\right)' = \frac{(p-1)z^{2p}+(p-2-p^2)z^p-1}{|(p-1)z^p-1|^2} < 0 \text{ and} \\ \Phi_U(z) &= \frac{1}{2}+\frac{1}{2}\left(\frac{U'(z)}{U''(z)}\right)' = \frac{p}{2}\frac{(p-1)z^{2p}-(p+1)z^p}{|(p-1)z^p-1|^2} < \frac{1}{2}. \end{aligned}$$

For $p \in (0,1), 1-p \leq A_U \leq 1$ *and* $\frac{1}{2}\frac{p}{p-1} \leq \Phi_U \leq 0$ *while for* $p < 0$*,* $1 \leq A_U \leq 1-p$ *and* $0 \leq \Phi_U \leq \frac{1}{2}\frac{p}{p-1} < \frac{1}{2}$*.*

ii) $U(z) = \frac{z^p}{p} + \frac{z^q}{q}, z > 0$*. Then for* $z > 0$ *and* $A(p,q) = p+q-2-(p-q)^2$ *one has*

$$\begin{aligned} A_U(z) &= \frac{(1-p)+(1-q)z^{q-p}}{1+z^{q-p}}, \quad \left(\frac{U'(z)}{U''(z)}\right)' = \frac{(p-1)+(q-1)z^{2(q-p)}+A(p,q)z^{q-p}}{|(p-1)+(q-1)z^{q-p}|^2} < 0, \\ \Phi_U(z) &= \frac{1}{2}\frac{p(p-1)+q(q-1)z^{2(q-p)}+(4pq-p^2-q^2-p-q)z^{q-p}}{|(p-1)+(q-1)z^{q-p}|^2} < \frac{1}{2}. \end{aligned}$$

Note that $1-q \leq A_U \leq 1-p$ *and* Φ_U *is bounded from below as the quotient of two polynomials of the same degree.*

3.4.1 Normed space of (L, Z^L, N^L) under exponential moments condition

In this section, we identify a normed space for the triplet (L, Z^L, N^L) under the following assumption:

Assumption 3.4.7. *For all* $r > 1$*, we have* $\mathbb{E}\left[\exp\left(r\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\right] < +\infty$*.*

Assumption 3.4.7 has been used in [FMW12, MW13] for the study of the Bellman BSDE (3.11) which corresponds to our main system of interest (3.9) in the power utility case. It also appears in papers studying BSDE with drivers satisfying the same properties as F_U described by Remark 3.4.5, see [MW12, BH08, DHR11]. Assumption 3.4.7 will enable us to apply the following existence and uniqueness result.

Theorem 3.4.8. *[MW12, Theorem 5, Corollary 1 and 2] Suppose that Assumption 3.4.7 holds and the filtration* $\mathbb{F}$ *is continuous. Let* F *be a driver satisfying the following conditions:*

i) F *is independent of* y *and for every* $t \in [0,T]$*, the map* $\mathbb{R}^n \ni z \mapsto F(t,z)$ *is continuous.*

ii) F *is locally Lipschitz in* z *with Lipschitz constant* $k > 0$*, i.e. for all* $(t, z_1, z_2) \in [0,T] \times \mathbb{R}^n \times \mathbb{R}^n$*,*

$$F(t,z_1) - F(t,z_2)| \leq k\left(||\sigma_t\mu_t|| + ||\sigma_t z_1|| + ||\sigma_t z_2||\right)\left(||\sigma_t(z_1-z_2)||\right).$$

iii) F *has quadratic growth in* z*, i.e. there exists* $\alpha, \gamma > 0$ *such that for all* $(t,z) \in [0,T] \times \mathbb{R}^n$*,*

$$|F(t,z)| \leq \alpha||\sigma_t\mu_t||^2 + \frac{\gamma}{2}||\sigma_t z||^2.$$

Then the following hold:
Existence*. The BSDE*$(F,0)$ *admits a solution* (Y,Z,N) *such that* $Y \in \varXi$*. Moreover,* $(Z,N) \in \mathcal{H}^r \times \mathcal{M}^r$ *for all* $r \geq 1$*.*
Comparison principle and uniqueness*. Let* F' *be another generator satisfying i), ii) and*

iii) and $(Y^{'}, Z^{'}, N^{'})$ a solution to BSDE($F^{'}$, 0) with $Y^{'} \in \Xi$. Assume that F is convex and for each $t \in [0, T]$

$$F(t, Y_t^{'}, Z_t^{'}) \le F^{'}(t, Y_t^{'}, Z_t^{'}).$$

Then for each $t \in [0, T]$, $Y_t \le Y_t^{'}$. In particular (Y, Z, N) is the unique solution to BSDE(F, 0) with $Y \in \Xi$ (see [MW12, Theorem 6 and Corollary 2]).

Before proceeding with our analysis of the integrability properties of the solution, let us point out the following consequence of Assumption 3.4.7 pertaining to the absence of arbitrage and the finiteness of the value function u.

Proposition 3.4.9. *Suppose that Assumption 3.4.7 holds. Let $Z^\mu = \mathcal{E}(-\mu \cdot M)$ and $\mathbb{Q}^\mu$ be the probability measure equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and given by*

$$d\mathbb{Q}^\mu / d\mathbb{P} := Z_T^\mu = \mathcal{E}(-\mu \cdot M)_T . \tag{3.29}$$

The measure $\mathbb{Q}^\mu$ belongs to $\mathcal{M}^e(S)$ and for every $r > 1$, we have $\mathbb{E}[(Z_T^\mu)^r + (1/Z_T^\mu)^r] < +\infty$. If additionally Assumptions 3.4.1 and 3.4.2 hold, then the value function u defined by (3.5) is finitely valued.

Proof. Clearly Z^μ is a local martingale and one verifies that $Z^\mu S$ is a local martingale. Assumption 3.4.7 implies that $\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\right] < +\infty$ and therefore Z^μ is a uniformly integrable martingale by Novikov's criteria (see [Kaz94, Corollary 1.1.]). Thus we have $\mathbb{Q}^\mu \in \mathcal{M}^e(S)$. Let $r > 1$. We only show that $\mathbb{E}[(Z_T^\mu)^r] < +\infty$ as the arguments for $\mathbb{E}[(1/Z_T^\mu)^r] < +\infty$ are similar. Applying Hölder's inequality and using the supermartingale property of $\mathcal{E}(-r^2\mu \cdot M)$, we obtain

$$\begin{aligned}
\mathbb{E}[(Z_T^\mu)^r] &= \mathbb{E}\left[\exp\left(-r^2\int_0^T \mu_s dM_s - \frac{1}{2}r^2\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\right)^{\frac{1}{r}}\right] \\
&= \mathbb{E}\left[\left(\mathcal{E}\left(-r^2\mu \cdot M\right)_T\right)^{\frac{1}{r}} \exp\left(\frac{1}{2}(r^3 - r)\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\right] \\
&\le \left(\mathbb{E}\left[\mathcal{E}\left(-r^2\mu \cdot M\right)_T\right]\right)^{\frac{1}{r}} \left(\mathbb{E}\left[\exp\left(\frac{r^2(r+1)}{2}\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\right)\right]\right)^{\frac{r-1}{r}} < +\infty.
\end{aligned}$$

Suppose that Assumptions 3.4.1 and 3.4.2 hold. Then by Remark 3.2.4, U satisfies $(G_{a,b,1})$ with a and b given by (3.23). By Lemma 2.3.8, to show that u is finitely valued, it suffices to show that for every $p \in (-\infty, 0) \cup (0, 1)$, the value function $\overline{u}_p$ corresponding to the power utility $U_p(z) = \frac{z^p}{p}, z > 0$ is finitely valued. Note that $\overline{u}_p \le 0$ for $p \in (-\infty, 0)$. As $\mathbb{E}\left[(1/Z_T^\mu)^r\right] < +\infty$ for every $r > 1$, we have that for $p \in (0, 1)$, $\mathbb{E}\left[(Z_T^\mu)^{\frac{p}{p-1}}\right] < +\infty$ and $\overline{u}_p$ is finitely valued by Remark 2.3.1. The proof is complete. □

In the following Theorem we establish the main integrability properties of L under Assumption 3.4.7 on which the subsequent results are built on. Fist we recall the function κ introduced in Section 2.6.2 to derive sufficient conditions for the boundedness of L:

$$\kappa : (0, +\infty) \ni z \mapsto z^2 + \frac{1}{2}z + z\sqrt{z(z+1)}. \tag{3.30}$$

Theorem 3.4.10. *Suppose that Assumptions 3.3.1, 3.4.1, 3.4.2 and 3.4.7 hold. Then $L \in \mathcal{S}^r$ for every $r > 1$. Moreover, if $\mathbb{F}$ is continuous, then $\log L \in \Xi$.*

Proof. By Remark 3.4.4, U satisfies $(G_{a,b,1})$ with a and b given by (3.23). We assume w.l.o.g. that $a \in (0,1)$ since for $a^{'} < a$, U satisfies $(G_{a^{'},b,1})$ as it satisfies $(G_{a,b,1})$.
We begin with the first assertion. Let $r > 1$. By Proposition 2.4.7, L belongs to $\mathcal{S}^r$ if $L(1-a)$ belongs to $\mathcal{S}^r$. So we only have to show that $L(1-a) \in \mathcal{S}^r$. Set $p = 1-a$ and assume that $r > \frac{1}{1-\sqrt{p}}$. Then by Lemma 2.6.8, we have for $t \in [0,T]$

$$L_t(p) \le \left(\mathbb{E}\left[\exp\left(\kappa(\frac{p}{1-p}) \int_t^T \mu_s^\top d\langle M\rangle_s \mu_s \right) \Big| \mathcal{F}_t \right] \right)^{1-\sqrt{p}} \le \widetilde{M}_t^{1-\sqrt{p}}, \tag{3.31}$$

where $\widetilde{M}$ is the martingale defined by $\widetilde{M}_t = \mathbb{E}\left[\exp\left(\kappa(\frac{p}{1-p}) \int_0^T \mu_s^\top d\langle M\rangle_s \mu_s \right) \Big| \mathcal{F}_t \right]$ for $t \in [0,T]$.
Using Assumption 3.4.7 on the exponential moments of all orders of $\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s$, one deduces from Doob's maximal inequalities that $\widetilde{M} \in \mathcal{S}^q$ for all $q > 1$. One sees from the inequality (3.31) and the fact that $\widetilde{M} \in \mathcal{S}^{r(1-\sqrt{p})}$ that $L(p) \in \mathcal{S}^r$. Since $\mathcal{S}^l \subseteq \mathcal{S}^{l'}$ for $l < l^{'}$, we have $L(p) \in \mathcal{S}^r$ for all $r > 1$.

For the second assertion, the idea of the proof is to bound $|\log L|$ by processes in Ξ. Our proof consists of two steps. Let $\alpha \in (0,a)$, $\rho \in (0, \frac{\alpha}{b})$ and $\gamma \in (0, \min\{\frac{1}{b}, 1\})$.
Step 1. We show that $(\log L)^+ = \max\{logL, 0\} \in \Xi$. By Lemma 3.2.5, we have for $t \in [0,T]$

$$L_t(-\alpha) \le L_t^\rho + L_t^{\frac{\alpha}{a}}. \tag{3.32}$$

Since $a \le b$, we have $\rho - \frac{\alpha}{a} < 0$ and $L^\rho + L^{\frac{\alpha}{a}} \le 2\max\{1, L^{\frac{\alpha}{a}}\}$. We deduce from (3.32) that

$$\frac{a}{\alpha}\left(\log L_t(-\alpha) - \log 2\right) \quad \le \quad (\log L_t)^+ \,, \;\; t \;\in [0,T]. \tag{3.33}$$

By Lemma 3.2.5, we have for $t \in [0,T]$, $L_t \le L_t^{1-\gamma} + L_t(1-a)$ and thus

$$\begin{aligned} \log L_t &\le \log 2 + \log\max\{L_t^{1-\gamma}, L_t(1-a)\} \\ &\le \log 2 + (1-\gamma)\,(\log L_t)^+ + \log L_t(1-a). \end{aligned} \tag{3.34}$$

Note that $L(1-a)$ is a supermartingale with lower bound 1 (see [Nut10, Lemma 3.5]). The map $\mathbb{R} \ni z \mapsto z^+ = \max\{z, 0\}$ is increasing and $\log L(1-a) \ge 0$. The right hand term of (3.34) is therefore positive and we deduce that

$$\gamma(\log L)^+ \quad \le \quad \log 2 + \log L(1-a). \tag{3.35}$$

Summing up (3.33) and (3.34), we obtain the following bound for $(\log L)^+$

$$\frac{a}{\alpha}\left(\log L(-\alpha) - \log 2\right) \quad \le \quad (\log L)^+ \le \frac{1}{\gamma}\left(\log 2 + \log L(1-a)\right). \tag{3.36}$$

By Lemma A.6 in [MW13], $|\log L(-\alpha)|, \log L(1-a) \in \Xi$. Thus (3.36) implies that $(\log L)^+ \in \Xi$.
Step 2. We provide suitable bounds for $|\log L|$. By (3.32) we have $L(-\alpha) \le L^\rho(1 + L^{\frac{\alpha}{a}-\rho})$. We recall that $\frac{\alpha}{a} - \rho > 0$. Using the properties of the function log and the latter inequality we see that

$$\log L(-\alpha) \quad \le \quad \log 2 + \rho\log L + (\frac{\alpha}{a} - \rho)(\log L)^+. \tag{3.37}$$

Putting together (3.34) and (3.37), we obtain

$$\frac{1}{\rho}\left(\log L(-\alpha) - \log 2 - (\frac{\alpha}{a} - \rho)(\log L)^+ \right) \le \log L \le \log 2 + (1-\gamma)\,(\log L)^+ + \log L(1-a).$$

Since $(\log L)^+, \log L(1-a), \log L(-\alpha) \in \Xi$, the above inequalities entail that $\log L \in \Xi$. $\square$

Remark 3.4.11.
- *Observe that we didn't use the equation (3.24) describing the dynamics of L in the proof of Theorem 3.4.10. In particular, Theorem 3.4.10 does not require U to be three times differentiable provided it satisfies the condition $(G_{a,b,1})$ for some positive constants a and b.*
- *Under the growth condition $(G_{a,b,1})$, it was shown in Proposition 2.4.2 that L is bounded from above for $a \geq 1$. In the latter case $L \in \mathcal{S}^r$ for all $r \geq 1$ irrespective of Assumption 3.4.7.*

Having established that L admits moment of all orders, we will now use the equation (3.24) describing its dynamics to provide some additional properties. The following proposition shows that L enjoys a supermartingale or submartingale property depending on the sign of the function Φ_U.

Proposition 3.4.12. *Suppose that Assumptions 3.3.7, 3.4.1 and 3.4.2 hold. Let (ξ^L, Π^L) be given by (3.26),*

$$M^L = Z^L \cdot M + N^L \text{ and } A^L = \int_0^{\cdot} L_{s-}\Phi_U(\widehat{X})\left(\mu + \xi^L\right)^\top d\langle M\rangle \left(\mu + \xi^L\right). \tag{3.38}$$

With a and b defined by (3.23), the following assertions hold:

A1. Assume that $a \geq 1$. Then L is a submartingale, $M^L \in BMO$ and $A^L \in \mathcal{S}^r$ for every $r \geq 1$.

A2. Assume that $b \leq 1$. Then L is a supermartingale. If additionally Assumption 3.4.7 hold, then for every $r \geq 1$, we have $M^L \in \mathcal{M}^r$ and $\xi^L \cdot M + \Pi^L \in \mathcal{M}^r$.

For the proof of A1., we need the following lemma concerning positive bounded submartingales whose proof can be found in [Nut12c, Lemmas 5.9 and 5.11].

Lemma 3.4.13. *Let $\delta > 0$ and Z a submartingale satisfying $0 \leq Z \leq \delta$. Then for every $\tau, \nu \in \mathcal{T}(\mathbb{F})$ with $\tau \leq \nu$, we have*

$$\mathbb{E}\left[[Z]_\nu - [Z]_\tau | \mathcal{F}_\tau\right] \leq \mathbb{E}\left[Z_\nu^2 - Z_\tau^2 | \mathcal{F}_\tau\right].$$

Let $Z = Z_0 + M^Z + A^Z$ be the canonical decomposition of Z with M^Z its martingale part. Then $M^Z \in BMO$. The last assertion is true if Z is a supermartingale [4].

Proof of Proposition 3.4.12. A1. By Remark 3.4.4, $a \geq 1$ implies that $\Phi_U \geq 0$. Thus $A^L \geq 0$, and we infer from the canonical decomposition (3.24) of L that it is is a local submartingale. Since for $r > 1$, $L \in \mathcal{S}^r$ by Theorem 3.4.10 or Remark 3.4.11, L is a true submartingale.
We will apply Lemma 3.4.13 to show the BMO property of M^L. As $L_T = 1$, the submartingale property of L entails that $L \leq 1$. The local martingale M being continuous, and orthogonal to N^L, we have

$$[L] = [M^L] = \int_0^{\cdot} (Z^L)^\top d\langle M\rangle Z^L + [N^L] \text{ and } \Delta\left[M^L\right] = \Delta\left[N^L\right] = |\Delta N^L|^2 = |\Delta L|^2 \leq 1.$$

Using the fact that $L \leq 1$, and applying Lemma 3.4.13, we obtain that for $\tau \in \mathcal{T}(\mathbb{F})$

$$\begin{aligned}\mathbb{E}\left[\left[M^L\right]_T - \left[M^L\right]_{\tau-} | \mathcal{F}_\tau\right] &= \mathbb{E}\left[\left[M^L\right]_T - \left[M^L\right]_\tau + \Delta\left[M^L\right]_\tau | \mathcal{F}_\tau\right] \\ &= \mathbb{E}\left[[L]_T - [L]_\tau + \Delta\left[M^L\right]_\tau | \mathcal{F}_\tau\right] \leq \mathbb{E}\left[L_T^2 - L_\tau^2 | \mathcal{F}_\tau\right] + 1 \leq 2.\end{aligned}$$

[4] If Z is a positive bounded supermartingale, then $\overline{Z} = ||Z||_\infty - Z$ is positive bounded submartingale with the martingale part $||Z||_\infty - M^Z$.

We deduce from the above bound that $M^L \in BMO$. Let $r \geq 1$. As the map $\mathbb{R}^+ \ni x \mapsto x^r \in \mathbb{R}^+$ is convex, of moderate growth[5] and L is a bounded submartingale, Theorem 3.2 in [LLP80] ensures that

$$\mathbb{E}\left[(A_T^L)^r\right] \leq (2r)^r \mathbb{E}\left[\sup_{t\in[0,T]} |L_t|^r\right] \leq (2r)^r.$$

The process L being increasing, the above inequality implies that $A^L \in \mathcal{S}^r$.

A2. We suppose that $b \leq 1$. Then $\Phi_U \leq 0$ by Remark 3.4.4 and as a result $A^L \leq 0$. Consequently, the canonical decomposition (3.24) confers to L a local supermartingale structure. The process L being positive, it is a true supermartingale.
We suppose additionally that Assumption 3.4.7 holds. Let $r \geq 1$. First we show that $M^L \in \mathcal{S}^r$. Theorem 3.4.10 ensures that $L \in \mathcal{S}^r$. Since the map $\mathbb{R}^+ \ni z \mapsto z^r \in \mathbb{R}^+$ is of moderate growth, we infer from [LLP80, Theorem 3.1] that there exists $\eta > 0$, such that

$$\mathbb{E}\left[(A_T^L)^r\right] \leq \eta \mathbb{E}\left[\sup_{t\in[0,T)} |L_t|^r\right] \leq \eta \mathbb{E}\left[\sup_{t\in[0,T]} |L_t|^r\right] < +\infty.$$

Hence, $A^L \in \mathcal{S}^r$. We deduce that $M^L = L - L_0 - A^L \in \mathcal{S}^r$ since $\mathcal{S}^r$ is a vector space. To conclude that $M^L \in \mathcal{M}^r$, we apply the Burkholder-Davis-Gundy (BDG) inequalities which entail that for some positive constant c_r we have

$$\mathbb{E}\left[\left[M^L\right]_T^{\frac{r}{2}}\right] \leq c_r \mathbb{E}\left[\sup_{t\in[0,T]} |M_t^L|^r\right] < +\infty. \tag{3.39}$$

We now show that $\xi^L \cdot M + \Pi^L \in \mathcal{M}^r$. The process L being a supermartingale with terminal value 1, we have $L \geq 1$. Consequently,

$$\begin{aligned}
\left[\xi^L \cdot M + \Pi^L\right] &= \int_0^{\cdot} \left(\frac{Z^L}{L_-}\right)^\top d\langle M\rangle \left(\frac{Z^L}{L_-}\right) + \int_0^{\cdot} \frac{1}{L_-^2} d\left[N^L\right] \\
&\leq \int_0^{\cdot} \left(Z^L\right)^\top d\langle M\rangle \left(Z^L\right) + \left[N^L\right] = \left[M^L\right].
\end{aligned}$$

The above inequality and (3.39) yield $\mathbb{E}\left[\left[\xi^L \cdot M + \Pi^L\right]_T^{\frac{r}{2}}\right] \leq \mathbb{E}\left[\left[M^L\right]_T^{\frac{r}{2}}\right] < +\infty$. We conclude that $\xi^L \cdot M + \Pi^L \in \mathcal{M}^r$. □

In general, it is not possible to infer the integrability properties of M^L or $\xi^L \cdot M + \Pi^L$ from those of L in a general filtration without the additional supermartingale/supermartingale property of L. However, for $\mathbb{F}$ continuous we can apply Theorem 3.4.8 from BSDEs theory to infer the desired properties.

Proposition 3.4.14. *Suppose that Assumptions 3.3.1, 3.4.1, 3.4.2, 3.4.7 hold and $\mathbb{F}$ is continuous. Let Z^L, N^L be as in Theorem 3.3.5, (ξ^L, Π^L) defined by (3.26) and ν the optimal trading strategy. For every $r \geq 1$, we have :*

i) $\xi^L \cdot M + \Pi^L \in \mathcal{M}^r$.

ii) $\nu \in \mathcal{H}^r$.

iii) $L \in \mathcal{S}^r$ *and* $M^L = Z^L \cdot M + N^L \in \mathcal{M}^r$.

[5] Let f be a right-continuous and increasing function defined on $\mathbb{R}^+$, and such that $f(0) = 0$ and $f(x) > 0$ for $x > 0$. f is said to be of *moderate growth*, if there exists $\gamma > 1$ such that $\sup_{x>0} \frac{f(\gamma x)}{f(x)} < +\infty$.

Proof. i) By (3.27), $(\log L, \xi^L, \Pi^L)$ is a solution to the BSDE$(F_U(\cdot,\widehat{X},\cdot),0)$, i.e.

$$d\log L_t \;=\; \xi^L_t dM_t + d\Pi^L_t - F_U(t,\widehat{X}_t,\xi^L_t)dK_t - \frac{1}{2}d\left[\Pi^L\right]_t,\; t\in[0,T],\; \log L_T = 0.$$

By Remark 3.4.5, $F_U(\cdot,\widehat{X},\cdot)$ is locally Lipschitz and has quadratic growth in z. Since $\log L \in \Xi$, we infer from Theorem 3.4.8 that $\xi^L\cdot M + \Pi^L \in \mathcal{M}^r$.
ii) By (3.13), $\nu = \frac{1}{A_U(\widehat{X})}\left(\mu+\xi^L\right)$. The first assertion and Assumption 3.4.7 guarantee that $\mu+\xi^L\in\mathcal{H}^r$. As $\frac{1}{A_U}\le\frac{1}{\underline{c}}$ by Assumption 3.4.1, we deduce that $\nu\in\mathcal{H}^r$.
iii) We have $L\in\mathcal{S}^r$ by Theorem 3.4.10. Let $A^L = \int_0^\cdot L\Phi_U(\widehat{X})\left(\mu+\xi^L\right)^\top d\langle M\rangle\left(\mu+\xi^L\right)$. We recall that $L = L_0 + M^L + A^L$. Since $\mu,\xi^L\in\mathcal{H}^r$ and Φ_U is bounded, we have $A^L\in\mathcal{S}^r$. As $\mathcal{S}^r$ is vector space, $M^L = L - L_0 - A^L \in\mathcal{S}^r$ and by Burkholder-Davis-Gundy inequalities, we have $M^L\in\mathcal{M}^r$. □

In the sequel, we assume that $\mathbb{F}$ is continuous. We want to exploit the canonical decomposition (3.27) of L to give a refinement of the estimates in Lemma 3.2.5 for U satisfying Assumptions 3.4.1 and 3.4.2, and a,b given by (3.23). We already noticed on the basis on Lemma 3.2.5 in Section 2.4 that L inherits the integrability properties of $L(1-a)$ while $L(1-b)$ inherits those of L (see for example Theorem 2.5.3). We will rely on the comparison principle for BSDEs to establish a more precise relationship between the processes $L(1-b), L$ and $L(1-a)$. First, let us introduce some notation applying to the power utility case $U(z)=\frac{z^p}{p}, z>0,\ p\in(-\infty,0)\cup(0,1)$. Fix $p\in(-\infty,0)\cup(0,1)$. By Theorem 3.3.5, $L(p)$ has the canonical decomposition

$$L(p) = L_0(p) + \int_0^\cdot Z^{L(p)}dM + N^{L(p)} + \frac{1}{2}\frac{p}{p-1}\int_0^\cdot\left(\mu+\frac{Z^{L(p)}}{L(p)}\right)^\top d\langle M\rangle\left(\mu+\frac{Z^{L(p)}}{L(p)}\right),$$

where $M^{L(p)} = \int_0^\cdot Z^{L(p)}dM + N^{L(p)}$ is the Kunita-Watanabe decomposition of its martingale part. Similarly as for (Z^L,N^L) , we consider the pair $(\xi^{L(p)},\Pi^{L(p)})$ defined by

$$\xi^{L(p)} = Z^{L(p)}/L(p) \text{ and } \Pi^{L(p)} = \int_0^\cdot\frac{1}{L(p)}dN^{L(p)}. \tag{3.40}$$

Now $L_T(p)=0$ and by Ito's formula,

$$d\log L(p) \;=\; \xi^{L(p)}_t dM_t + d\Pi^{L(p)}_t - f_p(t,\xi^{L(p)})dK_t - \frac{1}{2}\left[\Pi^{L(p)}\right]_t,\; t\in[0,T], \tag{3.41}$$

where for $(t,z)\in[0,T]\times\mathbb{R}^n$,

$$f_p(t,z) \;=\; -\frac{1}{2}\frac{p}{p-1}||\sigma_t(\mu_t+z)||^2 + \frac{1}{2}||\sigma_t z||^2. \tag{3.42}$$

Taking $p\in\{1-a,1-b\}$ and recalling that $\underline{\alpha} = \frac{1}{2}\frac{a-1}{a}$ and $\overline{\alpha} = \frac{1}{2}\frac{b-1}{b}$, we have

$$\begin{aligned} f_{1-a}(t,z) &= -\frac{1}{2}\frac{1-a}{-a}||\sigma_t(\mu_t+z)||^2 + \frac{1}{2}||\sigma_t z||^2 = -\underline{\alpha}||\sigma_t(\mu_t+z)||^2 + \frac{1}{2}||\sigma_t z||^2,\\ f_{1-b}(t,z) &= -\frac{1}{2}\frac{1-b}{-b}||\sigma_t(\mu_t+z)||^2 + \frac{1}{2}||\sigma_t z||^2 = -\overline{\alpha}||\sigma_t(\mu_t+z)||^2 + \frac{1}{2}||\sigma_t z||^2. \end{aligned}$$

By Assumption 3.4.2, $\underline{\alpha}\le\Phi_U\le\overline{\alpha}$. Using the definitions of f_{1-a}, f_{1-b} and F_U given by (3.25), we derive from the last inequality that

$$f_{1-b}(t,z)\le F_U(t,\widehat{X}_t,z)\le f_{1-a}(t,z),\;(t,z)\in[0,T]\times\mathbb{R}^n. \tag{3.43}$$

The comparison principle of Theorem 3.4.8 suggests that $\log L(1-b)\le\log L\le\log L(1-a)$. The following proposition gives the precise statement and a universal lower bound for L.

Proposition 3.4.15. *Suppose that Assumptions 3.3.1, 3.4.1, 3.4.2 and 3.4.7 hold, and $\mathbb{F}$ is continuous. Let $\mathbb{Q}^\mu$ be probability measure defined by (3.29) . Let a and b be defined by (3.23). The following assertions hold:*

i) The processes L,$L(1-a)$ and $L(1-b)$ satisfy

$$L(1-b) \leq L \leq L(1-a). \tag{3.44}$$

ii) For every $\tau \in \mathcal{T}(\mathbb{F})$, we have

$$\log L_\tau \geq -\frac{1}{2}\mathbb{E}^{\mathbb{Q}^\mu}\left[\int_\tau^T ||\sigma_s\mu_s||^2 dK_s \Big| \mathcal{F}_\tau\right]. \tag{3.45}$$

Proof. i) Let $p \in \{1-a, 1-b\}$ and $(\xi^{L(p)}, \Pi^{L(p)})$ be defined by (3.40). By Theorem 3.4.10, $\log L(p)$ and $\log L$ belong to $\varXi$. The drivers $F_U(\cdot, \widehat{X}, \cdot)$ and f_p being convex in z, Theorem 3.4.8 entails that $(\log L, \xi^L, \Pi^L)$ (resp. $(\log L(p), \xi^{L(p)}, \Pi^{L(p)})$) is the unique solution to the BSDE $(F_U(\cdot, \widehat{X}, \cdot), 0)$ (resp. $(f_p, 0)$ with $\log L \in \varXi$ (resp. $\log L(p) \in \varXi$). Moreover, due the inequalities (3.43), we have

$$\log L(1-b) \leq \log L \leq \log L(1-a).$$

One obtains (3.44) by taking the exponentials.
ii) Let $\tau \in \mathcal{T}(\mathbb{F})$. Using (3.27) describing the dynamical behavior of $\log L$, we have

$$\log L_\tau \quad = \quad -\int_\tau^T \xi_s^L dM_s - \int_\tau^T d\Pi_s^L + \int_\tau^T F(s, \widehat{X}_s, \xi_s^L) dK_s + \frac{1}{2}\int_\tau^T d[\Pi^L]_s. \tag{3.46}$$

As $\Phi_U < \frac{1}{2}$, we have for $s \in [0, T]$

$$F_U(s, \widehat{X}_s, \xi_s^L) \quad = \quad -\Phi_U(\widehat{X}_s)||\sigma_s(\mu_s + \xi_s^L)||^2 + \frac{1}{2}||\sigma_s\xi_s^L||^2 \geq -\frac{1}{2}||\sigma_s\mu_s||^2 - (\sigma_s\mu_s)^\top(\sigma_s\xi_s^L).$$

Inserting the above inequality in (3.46) and using the factorization $d\langle M\rangle = \sigma^\top\sigma dK$ yields

$$\log L_\tau \quad \geq \quad -\int_\tau^T \xi_s^L dR_s - \frac{1}{2}\int_\tau^T ||\sigma_s\mu_s||^2 dK_s - \int_\tau^T d\Pi_s^L + \frac{1}{2}\int_t^T d[\Pi^L]_s, \tag{3.47}$$

where $dR = dM + \langle M\rangle\mu$. We recall that $\frac{d\mathbb{Q}^\mu}{d\mathbb{P}} = Z_T^\mu$ on $\mathcal{F}_T$ where $Z^\mu = \mathcal{E}(-\mu \cdot M)$. By Girsanov's theorem, $\int_0^\cdot \xi^L dR$ is a $\mathbb{Q}^\mu$-local martingale with quadratic variation $\int_0^\cdot (\xi^L)^\top d\langle M\rangle\xi^L$. We have $\mathbb{E}\left[(Z_T^\mu)^2\right] < +\infty$ by Proposition 3.4.9. Using Proposition 3.4.14 and Hölder's inequality we obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\mu}\left[\int_0^T (\xi_s^L)^\top d\langle M\rangle_s \xi_s^L\right] \quad &= \quad \mathbb{E}\left[Z_T^\mu\left(\int_0^T (\xi_s^L)^\top d\langle M\rangle_s \xi_s^L\right)\right] \\
&\leq \quad \left(\mathbb{E}\left[(Z_T^\mu)^2\right]\right)^{\frac{1}{2}}\left(\mathbb{E}\left[\left(\int_0^T (\xi_s^L)^\top d\langle M\rangle_s \xi_s^L\right)^2\right]\right)^{\frac{1}{2}} < +\infty.
\end{aligned}$$

We infer from the Burkholder-Davis-Gundy (BDG) inequalities that $\int_0^\cdot \xi^L dR$ is a true $\mathbb{Q}^\mu$-martingale. As Π^L is orthogonal to M, Π^L is a $\mathbb{Q}^\mu$-local martingale. Arguing similarly as for $\int_0^\cdot \xi^L dR$ we obtain that Π^L is $\mathbb{Q}^\mu$-martingale. Hence taking conditional expectation w.r.t $\mathcal{F}_\tau$ in (3.47) we obtain

$$\log L_\tau \quad \geq \quad -\frac{1}{2}\mathbb{E}^{\mathbb{Q}^\mu}\left[\int_\tau^T ||\sigma_s\mu_s||^2 dK_s \Big| \mathcal{F}_\tau\right].$$

This completes the proof. $\square$

Remark 3.4.16. *In view of the dichotomy relation (3.44), it is clear that L inherits the integrability properties of $L(1-a)$ as mentioned above. Compare to the estimates of L given by Lemma 3.2.5, (3.44) and (3.45) are more precise and one sees directly that L is bounded away from 0 and ∞ if $\log L(1-a)$ is bounded and $\mu \cdot M$ is a BMO martingale. Moreover, the bounds are more explicit than those derived using Lemma 3.2.5, see Remark 3.4.22.*

We close this section by establishing an *a priori* estimate for $\log L$ in terms of the exponential moments of $\int_0^\cdot ||\sigma\mu||^2 dK$. The *a priori* estimates will further shed light on the BMO property of $\mu \cdot M$ for the boundedness of $\log L$ from a BSDE perspective. It relies on Theorem 3.4.10 and standard arguments from the theory of BSDEs with quadratic growth, see [MW12, Proposition 1]. We recall that for $x \in \mathbb{R}$, $\operatorname{sign}(x) = \frac{x}{|x|} 1_{\{x \neq 0\}}$.

Proposition 3.4.17. *Suppose that Assumptions 3.3.1, 3.4.1, 3.4.2 and 3.4.7 hold, and $\mathbb{F}$ is continuous. Let $\beta = \max\{|\underline{\alpha}|, |\overline{\alpha}|\}, \delta > 0, \gamma = 1 + 2\beta(1+\delta)$ and $\alpha = \beta(1+\frac{1}{\delta})$. Then for every $\tau \in \mathcal{T}(\mathbb{F})$, we have*

$$|\log L_\tau| \leq \frac{1}{\gamma} \log \left(\mathbb{E}\left[\exp\left(\gamma\alpha \int_\tau^T ||\sigma_s \mu_s||^2 dK_s \right) \Big| \mathcal{F}_\tau \right] \right). \tag{3.48}$$

For $\delta = \sqrt{\frac{1+2\beta}{2\beta}}$, we have $\gamma\alpha = \kappa(2\beta)$ where κ is defined by (3.30).

Proof. Let $\Theta = \gamma|\log L| + \gamma\alpha \int_0^\cdot ||\sigma_s\mu_s||^2 dK_s$. To show (3.48), it suffices to show that $\exp(\Theta)$ is submartingale. Recall that by (3.27), the dynamics of $\log L$ is described by the equation

$$d\log L_t = \xi_t^L dM_t + d\Pi_t^L - F_U(t, \widehat{X}_t, \xi_t^L)dK_t - \frac{1}{2} d\left[\Pi^L\right]_t, \quad t \in [0,T], \ \log L_T = 0.$$

Let $\Gamma^{\log L}$ be the local time of $\log L$. By Ito's formula, we have for $t \in [0,T]$

$$\begin{aligned} d\Theta_t &= \operatorname{sign}(\log L_t)\left(\gamma\xi_t^L dM_t + \gamma d\Pi_t^L - \gamma F_U(t, \widehat{X}_t, \xi_t^L)dK_t - \frac{\gamma}{2} d\left[\Pi^L\right]_t \right) + \\ &\quad + \gamma d\Gamma_t^{\log L} + \gamma\alpha ||\sigma_t\mu_t||^2 dK_t. \end{aligned}$$

We deduce that for $t \in [0,T]$, we have

$$\begin{aligned} d\exp(\Theta_t) &= \exp(\Theta_t)\left(\operatorname{sign}(\log L_t)\gamma\xi_t^L dM_t + \operatorname{sign}(\log L_t)\gamma d\Pi_t^L \right) \\ &\quad + \exp(\Theta_t)\left(-\operatorname{sign}(\log L_t)\gamma F_U(t, \widehat{X}_t, \xi_t^L) + \frac{1}{2}\gamma^2 ||\sigma_t\xi_t^L||^2 + \gamma\alpha||\sigma_t\mu_t||^2 \right) dK_t \\ &\quad + \gamma\exp(\Theta_t) d\Gamma_t^{\log L} + \frac{1}{2}\exp(\Theta_t)\left(\gamma^2 - \operatorname{sign}(\log L_t)\gamma \right) d\left[\Pi^L\right]_t. \end{aligned} \tag{3.49}$$

Since $\gamma > 1$, we have $\gamma^2 - \operatorname{sign}(\log L_t)\gamma \geq 0$. With the respective choices of γ and α, (3.28) reads

$$|F_U(t, \widehat{X}_t, \xi_t^L)| \leq \alpha||\sigma_t\mu_t|| + \frac{1}{2}\gamma||\sigma_t\xi_t^L||^2, \forall\, t \in [0,T],$$

and therefore

$$-\operatorname{sign}(\log L_t)\gamma F_U(t, \widehat{X}_t, \xi_t^L) + \frac{1}{2}\gamma^2||\sigma_t\xi_t^L||^2 + \gamma\alpha||\sigma_t\mu_t||^2 \geq 0, \forall\, t \in [0,T].$$

As $\Gamma^{\log L} \geq 0$, the above inequalities and (3.49) entail that $\exp(\Theta)$ is a local submartingale. Theorem 3.4.10 and Assumption 3.4.7 ensure that $\exp(\Theta)$ is uniform integrable. We deduce that $\exp(\Theta)$ is a true submartingale and (3.48) is a consequence of the submartingale property. ☐

Remark 3.4.18. *From (3.48), one sees that $\log L$ is bounded as soon as the* critical exponent *(see Definition 3.4.20 below) of $\mu \cdot M$ is strictly bigger than $\kappa(2\beta)$. The boundedness of $\log L$ is therefore determined by the critical exponent of $\mu \cdot M$ and the bounds of Φ_U.*

3.4.2 Normed space of the solution $(\widehat{X}, L, Z^L, N^L)$ under BMO condition

So far we have determined a suitable normed space for the triplet (L, Z^L, N^L) and the optimal trading strategy ν. It remains to identify a normed space for the optimal wealth process $\widehat{X} = x_0\mathcal{E}(\nu \cdot R)$ with R given by (3.2). The properties of ν given by Proposition 3.4.14 are in general not sufficient to embed $\widehat{X}$ into a suitable normed space. Recall that R is a $\mathbb{Q}^\mu$-local martingale and following the links between BMO martingales and reverse Hölder's inequalities, a BMO property of the $\mathbb{Q}^\mu$-local martingale $\nu \cdot R$ will be enough to assert that $\widehat{X} \in \mathcal{S}^k(\mathbb{Q}^\mu)$ for some $k > 1$. We know from Theorem 2.6.15 that the latter property is satisfied if $\mu \cdot M$ is a BMO martingale, and L is bounded away from 0 and ∞. Our goal in this section is to provide auxiliary properties for (Z^L, N^L) under BMO conditions ensuring the boundedness of L and give an alternative unique characterization of the solution $(\widehat{X}, L, Z^L, N^L)$.

We introduce the following concept of solution which also leads to the uniqueness of a solution to the system (3.9).

Definition 3.4.19. *A solution (X, l, Z^l, N^l) to (3.9) is said to be* bounded *if l is bounded away from 0 and ∞, or equivalently $\log l$ is bounded.*

Let us recall the notion of critical exponent of a continuous local martingale, see [Kaz94].

Definition 3.4.20. *Let N be a continuous local martingale with $N_0 = 0$. The* critical exponent *of N is the constant $b(N)$ defined by*

$$b(N) \quad := \quad \sup\left\{ l \geq 0 \mid \sup_{\varsigma \in \mathcal{T}(\mathbb{F})} \left|\left| \mathbb{E}\left[\exp\Big(l\left(\langle N\rangle_T - \langle N\rangle_\varsigma\right)\Big) | \mathcal{F}_\varsigma \right] \right|\right|_\infty < \infty \right\}.$$

By the John-Nirenberg inequality ([Kaz94, Theorem 2.2]), $b(N) > 0$ if and only if $N \in BMO$.

The following proposition giving sufficient conditions for $\log L$ to be bounded is a reformulation of Proposition 2.6.12. It builds on Proposition 3.4.15 to give more precise bounds of L than those in Proposition 2.6.12.

Proposition 3.4.21. *Suppose that U satisfies $(G_{a,b,1})$ with $a \leq b$. Let κ be defined by (3.30). We consider the following two cases:*

i) $a \geq 1$ and $\mu \cdot M \in BMO$

ii) $a \in (0, 1)$ and $b(\mu \cdot M) > \kappa\left(\frac{1-a}{a}\right)$.

If i) or ii) holds, then there exist two strictly positive constants $\underline{l}, \bar{l}$ depending only on a, b and $||\mu \cdot M||_{BMO}$ such that

$$\underline{l} \leq L \leq \bar{l}.$$

Suppose that Assumptions 3.3.1, 3.4.1, 3.4.2 and 3.4.7 are satisfied and $\mathbb{F}$ is continuous. Let a and b be given by (3.23). If i) or ii) holds, then there exists $\delta > 0$, depending only on $||\mu{\cdot}M||_{BMO}$ such that

$$\exp\left(-\delta\frac{1}{2}||\mu \cdot M||_{BMO}\right) \leq L \leq ||L(1-a)||_\infty. \tag{3.50}$$

Proof. The first assertion is a reformulation of Proposition 2.6.12. We only prove the second. Let $\mathbb{Q}^\mu$ be the probability measure equivalent to $\mathbb{P}$ with density process $Z^\mu = \mathcal{E}(-\mu \cdot M)$. As $\mu \cdot M \in BMO$, we infer from Theorem 3.2.6 that $R = \mu \cdot M + \int_0^\cdot d\langle M\rangle_s \mu_s \in BMO(\mathbb{Q}^\mu)$.

Moreover, $||R||_{BMO(\mathbb{Q}^\mu)} \leq \delta||\mu \cdot M||_{BMO}$ where δ is a constant depending only on $||\mu \cdot M||_{BMO}$. Since R has quadratic variation $\int_0^\cdot ||\sigma\mu||^2 dK$, we deduce that for every $\tau \in \mathcal{T}(\mathbb{F})$

$$\mathbb{E}^{\mathbb{Q}^\mu}\left[\int_\tau^T ||\sigma_s\mu_s||^2 dK_s \middle| \mathcal{F}_\tau\right] = \mathbb{E}^{\mathbb{Q}^\mu}\left[[R]_T - [R]_\tau | \mathcal{F}_\tau\right] \leq ||R||^2_{BMO(\mathbb{Q}^\mu)} \leq \delta^2 ||\mu \cdot M||^2_{BMO}.$$

Using the lower bound (3.45) and the above inequality, we obtain that for $\tau \in \mathcal{T}(\mathbb{F})$, we have

$$\log L_\tau \geq -\frac{1}{2}\mathbb{E}^{\mathbb{Q}^\mu}\left[\int_\tau^T ||\sigma_s\mu_s||^2 dK_s \middle| \mathcal{F}_\tau\right] \geq -\frac{1}{2}\delta^2||\mu \cdot M||^2_{BMO}.$$

To complete the proof, it suffices to show that $||L(1-a)||_\infty < +\infty$ and apply (3.44). If $a \geq 1$, then $L(1-a)$ is a submartingale with terminal value 1 by Proposition 3.4.12. Hence $||L(1-a)||_\infty = 1$. Now for $a \in (0,1)$, $\log L(1-a)$ satisfies (3.48) with $2\beta = \frac{1-a}{a}$. As a result, $||L(1-a)||_\infty < +\infty$ since $b(\mu \cdot M) > \kappa(\frac{1-a}{a})$. $\square$

Remark 3.4.22. *The hypotheses in Proposition 3.4.21 ensure that* $||\log L(1-a)||_\infty < +\infty$*, see Theorem 2.6.11. Suppose that* $a \in (0,1)$*. Following Lemma 3.2.5, for every* $\gamma \in (0, \min\{1, \frac{1}{b}\})$*, one can take* $\bar{l}$ *as the unique solution to the equation*

$$z = z^{1-\gamma} + ||L(1-a)||_\infty, \; z > 0. \tag{3.51}$$

Clearly $||L(1-a)||_\infty \leq ||L(1-a)||_\infty^{1-\gamma} + ||L(1-a)||_\infty$ *and thus* $||L(1-a)||_\infty \leq \bar{l}$ *by Lemma 2.4.1. The upper bound* $||L(1-a)||_\infty$ *is therefore more precise and sharper than* $\bar{l}$ *for all possible choices of* γ*. Note that the lower bound* $\exp\left(-\delta\frac{1}{2}||\mu \cdot M||_{BMO}\right)$ *is valid also for* $L(p)$ *for all* $p \in (-\infty, 0) \cup (0,1)$*. Using once more Lemma 3.2.5, for arbitrary* $\alpha \in (0,a)$ *and* $\rho \in (0, \frac{\alpha}{b})$ *one can choose* $\underline{l}$ *as follows*

$$\underline{l} = \left(\frac{\inf_{t\in[0,T]} L_t(-\alpha)}{1 + (1+\bar{l})^{\frac{\alpha}{a}-\rho}}\right)^{\frac{1}{\rho}} \quad \text{or } \underline{l} = \left(\frac{\exp\left(-\delta\frac{1}{2}||\mu \cdot M||_{BMO}\right)}{1 + (1+\bar{l})^{\frac{\alpha}{a}-\rho}}\right)^{\frac{1}{\rho}} \leq \exp\left(-\delta\frac{1}{2}||\mu \cdot M||_{BMO}\right).$$

One sees in the second case that the lower bound $\exp\left(-\delta\frac{1}{2}||\mu \cdot M||_{BMO}\right)$ *is more explicit and sharper than* $\underline{l}$*. For the case* $a \geq 1$*, one shows analogously that the bounds given by (3.50) are more sharper than* $\underline{l}$ *and* $\bar{l}$*. The canonical decomposition of* L *therefore leads to more precise bounds for* L*.*

The following theorem shows that under the same conditions as in Proposition 3.4.21, the quadruplet $(\widehat{X}, L, Z^L, N^L)$ is the unique bounded solution to the the system (3.9).

Theorem 3.4.23. *Suppose that Assumptions 3.3.1, 3.4.1 and 3.4.2 hold. Let* a, b *be defined by (3.23) and* $\mathbb{Q}^\mu$ *the probability measure given by (3.29). Let* Z^L *and* N^L *be as in (3.12) and* (ξ^L, Π^L) *be given by (3.26). We consider the following two cases:*

i) $a \geq 1$ *and* $\mu \cdot M \in BMO$*.*

ii) $a \in (0,1)$ *and* $b(\mu \cdot M) > \kappa(\frac{1-a}{a})$*.*

If i) or ii) holds, then we have the following assertions:

C1. There exists $C_{BMO} > 0$ *depending only on* a, b *and* $||\mu \cdot M||_{BMO}$ *such that*

$$||Z^L \cdot M + N^L||^2_{BMO} + ||\xi^L \cdot M||^2_{BMO} + ||\Pi^L||^2_{BMO} + ||\nu \cdot M||^2_{BMO} \leq C_{BMO}. \tag{3.52}$$

C2. There exist $k > 1$ *depending only* a, b *and* $||\mu \cdot M||_{BMO}$ *such that* $\widehat{X} \in \mathcal{S}^k(\mathbb{Q}^\mu)$ *and* $\widehat{Y} \in \mathcal{S}^k$*.*

C3. $(\widehat{X}, L, Z^L, N^L)$ *is the unique bounded solution to the system (3.9).*

Proof. Suppose that i) or ii) holds. From Remark 3.4.4, U satisfies $(G_{a,b,1})$ and Proposition 3.4.21 implies that there exists $\underline{l}$ and $\bar{l}$ depending only a, b and $||\mu \cdot M||_{BMO}$ such that

$$\underline{l} \leq L \leq \bar{l}.$$

C1. We follow similar ideas as in [Mor09b]. Let $(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{T}(\mathbb{F})$ be a localizing sequence for the local martingales $\xi^L \cdot M$, N^L and Π^L. Let $n \in \mathbb{N}$ and $\tau \in \mathcal{T}(\mathbb{F})$. An application of Itô's formula to $\log L$ yields

$$\begin{aligned} \log L_{T\wedge\sigma_n} - \log L_{\tau\wedge\sigma_n} &= \int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \frac{1}{L_{s-}} dL_s - \frac{1}{2}\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \frac{1}{L_{s-}^2} d[L]_s^c \\ &+ \sum_{\tau\wedge\sigma_n < s \leq T\wedge\sigma_n} \left\{ \log\left(1 + \frac{\Delta L_s}{L_{s-}}\right) - \frac{\Delta L_s}{L_{s-}} \right\}. \end{aligned}$$

Note that $\frac{\Delta L}{L} = -1 + \frac{L}{L_-} > -1$ and $\log\left(1 + \frac{\Delta L}{L_-}\right) - \frac{\Delta L}{L_-} \leq 0$. The above equation entails that

$$\log L_{T\wedge\sigma_n} - \log L_{\tau\wedge\sigma_n} \leq \int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \frac{1}{L_{s-}} dL_s - \frac{1}{2}\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \frac{1}{L_{s-}^2} d[L]_s^c. \tag{3.53}$$

We recall from (3.24) that $L_T = 1$ and

$$dL_t = Z_t^L dM_t + dN_t^L + L_{t-}\Phi_U(\widehat{X}_t)\left(\mu_t + \xi_t^L\right)^\top d\langle M\rangle_t \left(\mu_t + \xi_t^L\right), \; t \in [0, T]. \tag{3.54}$$

Since M is continuous and orthogonal to N^L, we have $[L] = \int_0^\cdot (Z_s^L)^\top d\langle M\rangle_s Z_s^L + \left[N^L\right]$ and

$$\begin{aligned} \int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \frac{1}{L_{s-}^2} d\,[L]^c &= \int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \frac{1}{L_{s-}^2} (Z_s^L)^\top d\langle M\rangle_s Z_s^L + \int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \frac{1}{L_{s-}^2} d\left[N^L\right]_s^c \\ &= \int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} (\xi_s^L)^\top d\langle M\rangle_s \xi_s^L + \left[\Pi^L\right]_{T\wedge\sigma_n}^c - \left[\Pi^L\right]_{\tau\wedge\sigma_n}^c. \end{aligned}$$

The stopped processes $\left(\xi^L \cdot M\right)_{\cdot\wedge\sigma_n}$ and $\left(\Pi^L\right)_{\cdot\wedge\sigma_n}$ being martingales, inserting the last two equations into (3.53), and taking conditional expectations w.r.t $\mathcal{F}_{\tau\wedge\sigma_n}$, we see that

$$\begin{aligned} \mathbb{E}\left[\log \frac{L_{T\wedge\sigma_n}}{L_{\tau\wedge\sigma_n}} \middle| \mathcal{F}_{\tau\wedge\sigma_n}\right] &\leq \mathbb{E}\left[\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \Phi_U(\widehat{X}_s)(\mu_s + \xi_s^L)^\top d\langle M\rangle_s(\mu_s + \xi_s^L) \middle| \mathcal{F}_{\tau\wedge\sigma_n}\right] \\ &-\frac{1}{2}\mathbb{E}\left[\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} (\xi_s^L)^\top d\langle M\rangle_s \xi_s^L - \frac{1}{2}\left[\Pi^L\right]_{T\wedge\sigma_n}^c + \frac{1}{2}\left[\Pi^L\right]_{\tau\wedge\sigma_n}^c \middle| \mathcal{F}_{\tau\wedge\sigma_n}\right]. \end{aligned} \tag{3.55}$$

By Assumption 3.4.2, $\Phi_U \leq \overline{\alpha} < \frac{1}{2}$. Let $\epsilon \in (0, 1)$ such that $\epsilon|\overline{\alpha}| + \overline{\alpha} - \frac{1}{2} < 0$. We define:

$$\beta := \frac{1}{2} - \overline{\alpha} - \epsilon|\overline{\alpha}|, C := \log \bar{l}/\underline{l} \text{ and } \gamma := \overline{\alpha} + \frac{|\overline{\alpha}|}{\epsilon} > 0. \tag{3.56}$$

Using Kunita-Watanabe's inequality and Young's inequality $xy \leq \frac{x^2}{2\epsilon} + \frac{\epsilon y^2}{2}, x, y \in \mathbb{R}$, we have

$$\begin{aligned} &\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \Phi_U(\widehat{X}_s)(\mu_s + \xi_s^L)^\top \langle M\rangle_s(\mu_s + \xi_s^L) \\ &\leq \overline{\alpha}\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \left[\mu_s^\top d\langle M\rangle_s \mu_s + 2\mu_s^\top d\langle M\rangle_s \xi_s^L + (\xi_s^L)^\top d\langle M\rangle_s \xi_s^L\right] \\ &\leq (\epsilon|\overline{\alpha}| + \overline{\alpha})\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} (\xi_s^L)^\top d\langle M\rangle_s \xi_s^L + \left(\overline{\alpha} + \frac{|\overline{\alpha}|}{\epsilon}\right)\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n} \mu_s^\top d\langle M\rangle_s \mu_s. \end{aligned}$$

Inserting the latter inequality into (3.55) and rearranging terms, we obtain

$$\beta\mathbb{E}\left[\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n}(\xi_s^L)d\langle M\rangle_s\xi_s^L+\frac{1}{2}\left(\left[\Pi^L\right]^c_{T\wedge\sigma_n}-\left[\Pi^L\right]^c_{\tau\wedge\sigma_n}\right)\Big|\mathcal{F}_{\tau\wedge\sigma_n}\right]\leq\mathbb{E}\left[C+\gamma\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n}\mu_s^\top d\langle M\rangle_s\mu_s\Big|\mathcal{F}_{\tau\wedge\sigma_n}\right] \tag{3.57}$$

where β,γ and C are given by (3.56). Choosing $\tau=0$ in (3.57) and taking the limit as $n\to+\infty$, we observe that

$$\mathbb{E}\left[\int_0^T(\xi_s^L)d\langle M\rangle_s\xi_s^L+\left[\Pi^L\right]^c_T\right]<+\infty.$$

As a result, for $\tau\in\mathcal{T}(\mathbb{F})$, taking the limit in (3.57) as $n\to+\infty$ and applying Hunt's Lemma [DM82b, Theorem V.45], we obtain that

$$\beta\mathbb{E}\left[\int_\tau^T(\xi_s^L)^\top d\langle M\rangle_s\xi_s^L\Big|\mathcal{F}_\tau\right]\leq C+\gamma||\mu\cdot M||^2_{BMO}. \tag{3.58}$$

In particular $||\xi^L\cdot M||^2_{BMO}\leq\frac{1}{\beta}\left(C+\gamma||\mu\cdot M||^2_{BMO}\right)$. We now proceed to give an upper bound for $||Z^L\cdot M+N^L||^2_{BMO}$. Let $(\sigma_n)_{n\in\mathbb{N}}\in\mathcal{T}(\mathbb{F})$ be defined as above. Let $\tau\in\mathcal{T}(\mathbb{F})$ and $n\in\mathbb{N}$. Applying Itô's formula to L^2 and using (3.54) together with the martingale property of $(Z^L\cdot M)_{\cdot\wedge\sigma_n}$ and $N^L_{\cdot\wedge\sigma_n}$, we have

$$\begin{aligned}
\mathbb{E}\left[[L]_{T\wedge\sigma_n}-[L]_{\tau\wedge\sigma_n}|\mathcal{F}_{\tau\wedge\sigma_n}\right] &= \mathbb{E}\left[L^2_{T\wedge\sigma_n}-L^2_{\tau\wedge\sigma_n}-2\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n}L_{s-}dL_s\Big|\mathcal{F}_{\tau\wedge\sigma_n}\right]\\
&= \mathbb{E}\left[L^2_{T\wedge\sigma_n}-L^2_{\tau\wedge\sigma_n}|\mathcal{F}_{\tau\wedge\sigma_n}\right]\\
&\quad -2\mathbb{E}\left[\int_{\tau\wedge\sigma_n}^{T\wedge\sigma_n}L^2_{s-}\Phi_U(\widehat{X}_s)(\mu_s+\xi_s^L)^\top d\langle M\rangle_s(\mu_s+\xi_s^L)\Big|\mathcal{F}_{\tau\wedge\sigma_n}\right]\\
&\leq \bar{l}^2+2\bar{l}^2||\Phi_U(\widehat{X})||_\infty||(\mu+\xi^L)\cdot M||^2_{BMO}.
\end{aligned}$$

Taking the limits in the above inequality and applying once more Hunt's lemma [DM82b, Theorem V.45], we obtain

$$\mathbb{E}\left[\int_\tau^T(Z_s^L)^\top d\langle M\rangle_sZ_s^L+\left[N^L\right]_T-\left[N^L\right]_\tau\Big|\mathcal{F}_\tau\right]\leq\bar{l}^2(1+2||\Phi_U(\widehat{X})||_\infty||(\mu+\xi^L)\cdot M||^2_{BMO}) \tag{3.59}$$

Clearly, as L is bounded, the jumps of L are bounded by $2\bar{l}$. Moreover, $\Delta L=\Delta N^L$ and $\Delta\left[N^L\right]=|\Delta N^L|^2\leq4\bar{l}^2$. Hence

$$\mathbb{E}\left[\int_\tau^T(Z_s^L)^\top d\langle M\rangle_sZ_s^L+\left[N^L\right]_T-\left[N^L\right]_{\tau-}\Big|\mathcal{F}_\tau\right]\leq\bar{l}^2(5+2||\Phi_U(\widehat{X})||_\infty||(\mu+\xi^L)\cdot M||^2_{BMO}) \tag{3.60}$$

Given that τ is arbitrary, (3.60) entails that $||Z\cdot M+N^L||^2_{BMO}\leq\bar{l}^2(5+2||\Phi_U(\widehat{X})||_\infty||(\mu+\xi^L)\cdot M||^2_{BMO})$. We now show that $\Pi^L\in BMO$. We recall that $\Pi^L=\int_0^\cdot\frac{1}{L_-}dN^L$. Thus $\Delta\Pi^L=\frac{\Delta N^L}{L_-}=\frac{\Delta L}{L_-}\leq\bar{l}/\underline{l}$, and $\Delta\left[\Pi^L\right]=|\Delta\Pi^L|^2\leq(\bar{l}/\underline{l})^2$. Employing the estimate (3.59), we have for $\tau\in\mathcal{T}(\mathbb{F})$

$$\begin{aligned}
\mathbb{E}\left[\left[\Pi^L\right]_T-\left[\Pi^L\right]_{\tau-}|\mathcal{F}_\tau\right] &= \mathbb{E}\left[\left[\Pi^L\right]_T-\left[\Pi^L\right]_\tau+\Delta\left[\Pi^L\right]_\tau|\mathcal{F}_\tau\right]\leq\mathbb{E}\left[\int_\tau^T\frac{1}{L^2_{s-}}d\left[N^L\right]\Big|\mathcal{F}_\tau\right]+(\bar{l}/\underline{l})^2\\
&\leq (1/\underline{l})^2\mathbb{E}\left[\left[N^L\right]_T-\left[N^L\right]_\tau|\mathcal{F}_\tau\right]+(\bar{l}/\underline{l})^2\\
&\leq 2(\bar{l}/\underline{l})^2(1+||\Phi_U(\widehat{X})||_\infty||(\mu+\xi^L)\cdot M||^2_{BMO}).
\end{aligned} \tag{3.61}$$

It remains to show that $\nu \cdot M \in BMO$. Since $A_U \geq a$ by (3.23), an application of Kunita-Watanabe's inequality yields for $\tau \in \mathcal{T}(\mathbb{F})$:

$$\begin{aligned} \mathbb{E}\left[\int_\tau^T (\nu_s)^\top \langle M\rangle_s \nu_s \Big| \mathcal{F}_\tau\right] &= \mathbb{E}\left[\int_\tau^T \frac{1}{A_U^2(\widehat{X}_s)}(\mu_s + \xi_s^L)^\top d\langle M\rangle_s(\mu_s + \xi_s^L)\Big|\mathcal{F}_\tau\right] \\ &\leq \frac{2}{a^2}(||\mu \cdot M||^2_{BMO} + ||\xi^L \cdot M||^2_{BMO}) \end{aligned} \tag{3.62}$$

which proves the BMO property of $\nu \cdot M$. Combining (3.58), (3.60), (3.61) and (3.62), one obtains (3.52) with

$$C_{BMO} = 5\bar{l}^2 + 2(\bar{l}/\underline{l})^2 + \eta\frac{C}{\beta} + \left(2\eta + \frac{\gamma}{\beta}\right)||\mu \cdot M||^2_{BMO},$$

where $\eta = 4\bar{l}^2||\Phi_U(\widehat{X})||_\infty + 4(\bar{l}/\underline{l})^2||\Phi_U(\widehat{X})||_\infty + \frac{2}{a^2}$ and γ, C, β are given by (3.56).

C2. Recall that $\widehat{X} = x\mathcal{E}(\nu \cdot R)$ where $dR = dM + d\langle M\rangle\mu$. Note that

$$\nu \cdot R = \nu \cdot M - \langle \nu \cdot M, -\mu \cdot M\rangle.$$

Since $\nu \cdot M$ and $-\mu \cdot M$ are BMO martingales, $\nu \cdot R \in BMO(\mathbb{Q}^\mu)$ and Theorem 3.2.6 implies that $\widehat{X}$ is a $\mathbb{Q}^\mu$-uniformly integrable martingale. By Theorem 3.2.6, we have $||\nu \cdot R||_{BMO(\mathbb{Q}^\mu)} \leq C_3||\nu \cdot M||_{BMO}$ where C_3 is a constant depending only on $||\mu \cdot M||_{BMO}$. We deduce from assertion C1. and Corollary 2.2.8 that there exists k_1 depending only a, b and $||\mu \cdot M||_{BMO}$ such that $\widehat{X} \in \mathcal{S}^{k_1}(\mathbb{Q}^\mu)$.

Regarding $\widehat{Y} = \widehat{Y}_0\mathcal{E}\left(-\mu \cdot M + \Pi^L\right)$, we employ the same arguments. First observe that we have $\underline{l}/\bar{l} \leq 1 + \Delta\Pi^L = 1 + \frac{\Delta L}{L_-} \leq \bar{l}/\underline{l}$, and from the assertion C1., $||-\mu \cdot M + \Pi^L||^2_{BMO} \leq ||\mu \cdot M||^2 + C_{BMO}$. Hence, by Corollary 2.2.8 there exists $k_2 > 1$ depending only on a, b and $||\mu \cdot M||_{BMO}$ such that $\widehat{Y}$ belongs to $\mathcal{S}^{k_2}$. Take $k = \min\{k_1, k_2\}$.

C3. As $\log L$ is bounded by Proposition 3.4.21, $(\widehat{X}, L, Z^L, N^L)$ is a bounded solution to (3.9). Let (X, P, Q, N) be another bounded solution to (3.9). To show that $(\widehat{X}, L) = (X, P)$, it suffices to show that (X, P) has the martingale property, i.e. $XU^{'}(X)P$ is a uniformly integrable martingale. This will guarantee by Theorem 3.3.5 that $(X, P) = (\widehat{X}, L)$. To simplify notation, we set

$$\xi = \frac{Q}{P_-} \text{ and } \Pi = \int_0^\cdot \frac{1}{P_-}dN.$$

As (X, P) solves (3.9), $P_T = 1$ and

$$dP_t = Q_t dM_t + dN_t + P_{t-}\left(\mu_t + \xi_t\right)^\top d\langle M\rangle(\mu_t + \xi_t), t \in [0, T].$$

Since the equation describing the dynamical behavior of P has the same structure as (3.54), using similar arguments as in assertion C1.,the boundedness of $\log P$ entails that there exists $\delta > 0$ such that

$$||Q \cdot M + N||^2_{BMO} + ||\xi \cdot M + \Pi||^2_{BMO} < \delta.$$

Applying Itô's product rule, we have for $t \in [0, T]$

$$dU^{'}(X_t) = -U^{'}(X_t)\left[\left(\mu_t + \xi_t\right)dM_t + \left(\mu_t - \frac{1}{2}\frac{U^{(3)}(X_t)U^{'}(X_t)}{|U^{''}(X_t)|^2}(\mu_t + \xi_t)\right)^\top d\langle M\rangle_t(\mu_t + \xi_t)\right],$$

$$d(U^{'}(X_t)P_t) = U^{'}(X_t)P_t\left(-\mu_t dM_t + d\Pi_t\right),$$

and

$$d\left(X_tU^{'}(X_t)P_t\right) \;=\; X_tU^{'}(X_t)P_t\left[\left(-\mu_t - \frac{U^{'}(X_t)}{X_tU^{''}(X_t)}(\mu_t+\xi_t)\right)dM_t + d\Pi_t\right].$$

Let $\widetilde{N}$ be the local martingale starting at 0 and satisfying

$$d\widetilde{N}_t = \left(-\mu_t - \frac{U^{'}(X_t)}{X_tU^{''}(X_t)}(\mu_t+\xi_t)\right)dM_t + d\Pi_t, t\ \in[0,T].$$

As P is bounded away from 0 and ∞, there exists a constant $h>0$ such that $\frac{P}{P_-}\geq h$. Due to the continuity of M, the jumps of P are carried by N and the jumps $\Delta\widetilde{N}$ of $\widetilde{N}$ satisfy

$$1+\Delta\widetilde{N} \;=\; 1+\Delta\Pi = 1+\frac{P-P_-}{P_-} = \frac{P}{P_-}\geq h. \tag{3.63}$$

By (3.23), $-\frac{U^{'}(X)}{XU^{''}(X)}\leq\frac{1}{a}$. Thus $\widetilde{N}$ is a BMO martingale as the sum of BMO martingales. Moreover, as $1+\Delta\widetilde{N}\geq h$, we infer from [ISS79, Theorem 2] that $XU^{'}(X)P = xU^{'}(x_0)P_0\mathcal{E}(\widetilde{N})$ is a uniformly integrable martingale and thus (X,P,Q,N) is a solution to (3.9) with the martingale property. □

Remark 3.4.24. *Even if Φ_U is bounded and Lipschitz continuous, the driver F_U is not Lipschitz in x since it contains the product $\Phi_U(x)||\sigma(\mu+z)||^2$ and neither is the volatility or the drift of $\widehat{X}$. Moreover, note that μ is not required to be bounded. Thus we cannot expect to obtain the integrability condition $\widehat{X}\in\mathcal{S}^k(\mathbb{Q}^\mu)$ or to prove the existence and uniqueness of a bounded solution with standard techniques (see [LL17, MWZZ15, FI13, AH06, Del02]). This is the reason why we rely only on the duality result given in Theorem 3.2.2, the concept of solution with the martingale property, and Proposition 3.4.21. Theorems 3.3.5 and 3.4.23 give the existence of suitably integrable solutions to a class of fully coupled non standard FBSDEs.*

An important feature in Theorem 3.4.23 is the fact that the norms $||\xi^L\cdot M+\Pi^L||_{BMO}$ can be controlled through the parameters of the function U and $||\mu\cdot M||_{BMO}$. We will exploit this feature in Section 3.5 to show that for a family of utility functions for which the relative risk aversion are uniformly bounded, one can control the corresponding norms for the opportunity processes.

3.5 Risk aversion asymptotics

Throughout this section, we suppose that $\mathbb{F}$ is continuous. Having obtained explicit structure of the optimizers and studied their integrability properties, we investigate in this section their asymptotic behavior as the relative risk aversion coefficient approaches a constant $c\in(0,+\infty)$ or $+\infty$. To this end, we consider a sequence of utility functions $(U_m)_{m\in\mathbb{N}}$ satisfying the following conditions:

Assumption 3.5.1. *H1) For each $m\in\mathbb{N}$, U_m is three times continuously differentiable. For $z>0$, let*

$$A_{U_m}(z) := -\frac{zU_m^{''}(z)}{U_m^{'}(z)} \text{ and } \Phi_{U_m}(z) := 1-\frac{1}{2}\frac{U_m^{(3)}(z)U_m^{'}(z)}{|U_m^{''}(z)|^2} = \frac{1}{2}+\frac{1}{2}\left(\frac{U_m^{'}}{U_m^{''}}\right)^{'}(z)$$

H2) There exists $c_m>0$ such that $A_{U_m}\geq c_m$.

H3) There exists $\underline{\alpha}_m, \overline{\alpha}_m \in (-\infty, \frac{1}{2})$ *such that* $\underline{\alpha}_m \leq \Phi_{U_m} \leq \overline{\alpha}_m$.

H4) There exists $c \in (0, +\infty)$ *or* $c = +\infty$ *such that uniformly in* $z > 0$, *we have*

$$\lim_{m\to+\infty} \Phi_{U_m}(z) = \phi_{1-c} = \begin{cases} \frac{1}{2}\frac{c-1}{c} & \text{if } c \neq +\infty, \\ \frac{1}{2} & \text{else.} \end{cases} \tag{3.64}$$

The following remark resumes some consequences of Assumption 3.5.1.

Remark 3.5.2. *Let* $(U_m)_{m\in\mathbb{N}}$ *satisfying Assumption 3.5.1. W.l.o.g. we can assume that for each* $m \in \mathbb{N}$: $\underline{\alpha}_m := \inf_{x>0} \Phi_{U_m}(x)$ *and* $\overline{\alpha}_m := \sup_{x>0} \Phi_{U_m}(x)$. *Note that as* $(\Phi_{U_m})_{m\in\mathbb{N}}$ *converges uniformly, we have* $\lim_{m\to+\infty} \overline{\alpha}_m = \lim_{m\to+\infty} \underline{\alpha}_m = \lim_{m\to+\infty} \Phi_{U_m}$. *Following Remark 3.4.3, for each* $m \in \mathbb{N}$

$$a_m = \frac{1}{1-2\underline{\alpha}_m} \leq A_{U_m} \leq \frac{1}{1-2\overline{\alpha}_m} = b_m. \tag{3.65}$$

i) Let $c \in (0, +\infty)$ *or* $c = +\infty$ *such that (3.64) holds. Then uniformly in* $z > 0$, *we have*

$$\lim_{m\to+\infty} A_{U_m} = c. \tag{3.66}$$

The convergence of the sequences $(\underline{\alpha}_m)_{m\in\mathbb{N}}$ *and* $(\overline{\alpha}_m)_{m\in\mathbb{N}}$ *to the same limit* ϕ_{1-c} *and the bounds (3.65) imply that* $(A_{U_m})_{m\in\mathbb{N}}$ *converges uniformly. Moreover, we have*

$$\lim_{m\to+\infty} A_{U_m} = \frac{1}{1 - \lim_{m\to+\infty} \Phi_{U_m}} = c.$$

ii) There exists $a, b \in (0, +\infty)$ *with* $a \leq b$ *such that for every* $m \in \mathbb{N}$, U_m *satisfies* $(G_{a,b,1})$. *Indeed, for each* $m \in \mathbb{N}$, *the inequality (3.65) and Remark 3.2.4 imply that* U_m *satisfies* $(G_{a_m,b_m,1})$. *Due to the convergence of* $(\underline{\alpha}_m)_{m\in\mathbb{N}}$ *and* $(\overline{\alpha}_m)_{m\in\mathbb{N}}$, *the sequences* $(a_m)_{m\in\mathbb{N}}$ *and* $(b_m)_{m\in\mathbb{N}}$ *converge and are therefore bounded. Choosing* $a = \inf_{l\in\mathbb{N}} a_l$ *and* $b = \sup_{l\in\mathbb{N}} b_l$, *for* $m \in \mathbb{N}$, U_m *satisfies* $(G_{a,b,1})$ *since* $a \leq a_m$ *and* $b \geq b_m$.

iii) There exist $k_1 > 0$ *and* $k_2 \in \mathbb{R}$ *such that for each* $z > 0$, *we have the following convergence*

$$\lim_{m\to+\infty} U_m(z) = \begin{cases} k_1 \frac{z^{1-c}}{1-c} + k_2 & \text{if } c \in (0,1) \cup (1, +\infty), \\ k_1 \log z + k_2 & \text{if } c = 1. \end{cases} \tag{3.67}$$

Indeed, by Remark 3.4.3, H2) and H3) imply that for each $m \in \mathbb{N}$, $\frac{U'_m(0)}{U''_m(0)} = 0$. *Noting that for* $m \in \mathbb{N}$, $\frac{U''_m}{U'_m} = \left(\log U'_m\right)'$ *and applying [Rud76, Theorem 7.17] on uniform convergence and integration, together with the locally Lipschitz property of the exponential function, we obtain the desired pointwise convergence from the hypothesis H4).*

In view of assertion i) of Remark 3.5.2, the condition (3.64) can be considered as an asymptotic statement on the relative risk aversion of the investor. This is precisely the case if for each $m \in \mathbb{N}$, there exists $\gamma_m \in (0, +\infty)$ such that $A_{U_m}(z) = -\frac{zU''_m(z)}{U'_m(z)} = \gamma_m, z > 0$. Indeed in this case, we have $\Phi_{U_m} = \frac{1}{2} + \frac{1}{2}\left(\frac{U'_m}{U''_m}\right)' = \frac{1}{2}\frac{\gamma_m - 1}{\gamma_m}$. Clearly $\gamma_m \to c \in (0, +\infty]$ if and only if (3.64) holds. We will now give examples of sequences satisfying (3.64) and (3.66):

Example 3.5.3. *E1)* $U_m(z) = \frac{z^{p_m}}{p_m}, z > 0$ *where* $(p_m)_{m\in\mathbb{N}}$ *is a sequence in* $(-\infty, 0) \cup (0, 1)$ *converging to* $1 - c, c \in (0, +\infty)$ *or* $c = +\infty$.

E2) $U_m(z) = \log z + \frac{z^p_m}{p_m}, z > 0$, *where* $(p_m)_{m\in\mathbb{N}}$ *is a sequence in* $(-\infty, 0) \cup (0, 1)$ *converging to* 0. *Using the computations in Example 3.4.6, one has* $|A_{U_m} - 1| \leq p_m \to 0$, *and also* $|\Phi_{U_m}| \leq \frac{1}{2}|\frac{p_m}{1-p_m}| \to 0$.

E3) $U_m(z) = \delta e^{-mz} z^{-m}, z > 0$ *and* $\delta \leq 0$. *Then for* $z > 0$

$$U_m^{'}(z) = -m\delta e^{-mz} z^{-m-1}(1+z) \text{ and } U_m^{''}(z) = m\delta e^{-mz} z^{-m-2}\left[mz^2 + 2mz + m + 1\right],$$

$$A_{U_m}(z) = \frac{mz^2 + 2mz + m + 1}{1+z} \text{ and } \left(\frac{U_m^{'}(z)}{U_m^{''}(z)}\right)^{'} = -\frac{mz^2 + 2z(m+1) + m + 1}{|mz^2 + 2mz + m + 1|^2}.$$

Uniformly in z, *we have* $A_{U_m}(z) \to +\infty$ *and* $\Phi_{U_m}(z) = \frac{1}{2} + \frac{1}{2}\left(\frac{U_m^{'}}{U_m^{''}}\right)^{'}(z) \to \frac{1}{2}$.

We denote by $(u_m)_{m\in\mathbb{N}}$ and $(v_m)_{m\in\mathbb{N}}$, the primal and dual value functions associated to $(U_m)_{m\in\mathbb{N}}$. We recall that for $m \in \mathbb{N}$ and $x, y > 0$

$$u_m(x) = \sup_{\pi\in\mathcal{A}(x)} \mathbb{E}\left[U_m(X_T^\pi)\right] \text{ and } v_m(y) = \inf_{Y\in\mathcal{Y}} \mathbb{E}\left[V_m(yY_T)\right], \tag{3.68}$$

where V_m is the convex conjugate of U_m. Assuming that for each $m \in \mathbb{N}$, $u_m < +\infty$, we denote respectively by $\nu^m, \widehat{X}^m, \widehat{Y}^m$ and L^m, the optimal trading strategy, optimal wealth process , dual optimizer and generalized opportunity process corresponding to the primal and dual problems (3.68) with initial value $x = x_0 > 0$ and $y = u_m^{'}(x_0)$. We recall that for each $m \in \mathbb{N}, L^m := \frac{\widehat{Y}^m}{U_m^{'}(\widehat{X}^m)}$ is a special semimartingale and by Theorem 3.3.5, it admits the canonical decomposition

$$dL_t^m = Z_t^{L^m} dM_t + dN_t^{L^m} + L_t^m \Phi_{U_m}(\widehat{X}_t^m)\left(\mu_t + \xi_t^{L^m}\right)^\top d\langle M\rangle_t \left(\mu_t + \xi_t^{L^m}\right), \; t \in [0, T], \tag{3.69}$$

where $M^{L^m} = \int_0^\cdot Z^{L^m} dM + N^{L^m}$ is the Kunita-Watanabe decomposition of its martingale part while

$$\xi^{L^m} = \frac{Z^{L^m}}{L^m} \text{ and } \Pi^{L^m} = \int_0^\cdot \frac{1}{L^m} dN^{L^m}.$$

An application of Itô's formula to $\log L^m$ shows that $(\log L^m, \xi^{L^m}, \Pi^{L^m})$ is a solution to the BSDE $(F_{U_m}(\cdot, \widehat{X}^m, \cdot), 0)$ where for $(t, \omega, x, z) \in [0, T] \times \Omega \times (0, +\infty) \times \mathbb{R}^n$

$$F_{U_m}(t, \omega, x, z) := -\Phi_{U_m}(x)||\sigma_t(\omega)(\mu_t(\omega) + z)||^2 + \frac{1}{2}||\sigma_t(\omega)z||^2. \tag{3.70}$$

The explicit formulas of the optimizers are given by

$$\nu^m = \frac{1}{A_{U_m(\widehat{X}_m)}}\left(\mu + \xi^{L^m}\right), \widehat{X}^m = x_0\mathcal{E}\left(\nu^m \cdot R\right) \text{ and } \widehat{Y}^m = \widehat{Y}_0^m \mathcal{E}\left(-\mu \cdot M + \Pi^{L^m}\right). \tag{3.71}$$

Our goal in the sequel is to study for $(U_m)_{m\in\mathbb{N}}$ satisfying Assumption 3.5.1, the convergence of the sequences $(\nu^m)_{m\in\mathbb{N}}, (\widehat{X}^m)_{m\in\mathbb{N}}$ and $(\widehat{Y}^m)_{m\in\mathbb{N}}$. Let us provide some intuition on the procedure we will adopt for our study. One observes from (3.71) that the convergence of the optimizers is entirely determined by the convergence of the sequences $(\xi^{L^m})_{m\in\mathbb{N}}$ and $(\Pi^{L^m})_{m\in\mathbb{N}}$ which are related to the martingale parts of $(\log L^m)_{m\in\mathbb{N}}$. Our study therefore reduces to the convergence of the sequence $(\log L^m)_{m\in\mathbb{N}}$ and its corresponding martingale parts. Recall that $(\log L^m, \xi^{L^m}, \Pi^{L^m})$ is a solution to the BSDE $(F_{U_m}(\cdot, \widehat{X}^m, \cdot), 0)$ and for every $(t,\omega,x,z) \in [0,T]\times\Omega\times(0,+\infty)\times\mathbb{R}^n$, (3.64) implies that

$$F_{U_m}(t,\omega,x,z) \longrightarrow f(t,\omega,z) = \begin{cases} f_{\log}(t,\omega,z) \text{ if } c=1, \\ f_{1-c}(t,\omega,z) \text{ if } c\in(0,1)\cup(1,+\infty), \\ f_{\exp}(t,\omega,z) \text{ else} \end{cases}$$

where for $(t,\omega,z)\in[0,T]\times\Omega\times\mathbb{R}^n$, $f_{\log}, f_{1-c}$ and $f_{\exp}$ are defined as follows[6]:

$$f_{\log}(t,\omega,z) = \frac{1}{2}||\sigma_t(\omega)z||^2, \quad f_{1-c}(t,\omega,z) = -\frac{1}{2}\frac{c-1}{c}||\sigma_t(\omega)(\mu_t(\omega)+z)||^2 + \frac{1}{2}||\sigma_t(\omega)z||^2, \tag{3.72}$$

$$f_{\exp}(t,\omega,z) = -\frac{1}{2}||\sigma_t(\omega)\mu_t(\omega)||^2 - (\sigma_t(\omega)\mu_t(\omega))^\top(\sigma_t(\omega)z). \tag{3.73}$$

As the drivers $F_{U_m}\to f\in\{f_{\log}, f_{1-c}, f_{\exp}\}$, one expects $\left(\log L^m, \xi^{L^m}, \Pi^{L^m}\right)_{m\in\mathbb{N}}$ to converge to the solution of the BSDE$(f,0)$ and $(\nu^m)_{m\in\mathbb{N}}, (\widehat{X}^m)_{m\in\mathbb{N}}$ and $(\widehat{Y}^m)_{m\in\mathbb{N}}$ to converge accordingly using (3.71). Typically the mode of convergence will depend on the integrability properties of the optimizers and the market price of risk μ. For the latter, we will consider two different assumptions. The first will be Assumption 3.4.7 on the exponential moments of all orders for $\int_0^T ||\sigma_s\mu_s||^2 dK_s$, and in this case, we will give the convergence of the optimal wealth processes and dual optimizers in the semimartingale topology $\mathcal{S}_0$. The second will be a BMO condition on $\mu\cdot M$ and we will be mainly interested in the convergence of the dual optimizers in entropy (see below). We briefly recall the concept of relative entropy between probability measures (see [Csi75]).

Definition 3.5.4. *Let $\mathbb{Q}_1, \mathbb{Q}_2$ be two probability measures. Then the relative entropy of $\mathbb{Q}_1$ w.r.t. $\mathbb{Q}_2$ denoted by $\mathcal{I}(\mathbb{Q}_1,\mathbb{Q}_2)$ is defined as*

$$\mathcal{I}(\mathbb{Q}_1,\mathbb{Q}_2) = \begin{cases} \mathbb{E}^{\mathbb{Q}_2}\left[\frac{d\mathbb{Q}_1}{d\mathbb{Q}_2}\log\frac{d\mathbb{Q}_1}{d\mathbb{Q}_2}\right] \textit{ if } \mathbb{Q}_1 \ll \mathbb{Q}_2, \\ +\infty \textit{ otherwise.} \end{cases} \tag{3.74}$$

A sequence of probability measures $(\mathbb{Q}^m)_{m\in\mathbb{N}}$ converges in entropy to a probability measure $\mathbb{Q}$ if

$$\lim_{m\to+\infty} \mathcal{I}(\mathbb{Q}^m,\mathbb{Q}) = 0.$$

Note that the drivers $f_{\log}$ and $f_{1-c}, c\in(0,1)\cup(1,+\infty)$ are identical to F_{U_m} with U_m replaced respectively by log and $U(z)=\frac{z^{1-c}}{1-c}, z>0$. Due to the pointwise convergence of $(U_m)_{m\in\mathbb{N}}$ given by Remark 3.5.2, the limits of the optimizers coincide with the optimizers of the respective utilities, see Remark 3.5.5. Contrary to $f_{\log}, f_{1-c}$, the driver $f_{\exp}$ is related to the exponential utility maximization problem, see [HIM05, Mor09a] or Section 3.5.3. Nevertheless, the limits of the optimizers will be related to the optimizers of the latter problem. We state separately the results for the cases $c\in(0,1)\cup(1,+\infty), c=1$ and $c=+\infty$ as they are connected respectively to the power, logarithmic and exponential utilities. However, the proof for all three cases are similar due to the identical structure of the drivers $f_{1-c}, f_{\log}$ and $f_{\exp}$. We will give a rigorous proof for $c\in(0,1)$ and adapt it accordingly in the other cases.

Remark 3.5.5. *Assume that $c\in(0,+\infty)$ and Assumption 3.4.7 holds. Then with $Z^\mu=\mathcal{E}(-\mu\cdot M)$, for every $y>0$, $\left(Z_T^\mu V_m^+(yZ_T^\mu)\right)_{m\in\mathbb{N}}$ is uniformly integrable, in particular $(U_m)_{m\in\mathbb{N}}$ satisfies the condition (UI) (see Definition 2.7.2). As the sequence $(U_m)_{m\in\mathbb{N}}$ converges pointwise and the condition $(G_{a,b,1})$ holds for each $m\in\mathbb{N}$ (see Remark 3.5.2), the condition (UI) follows from*

[6] In the sequel, we omit the dependence on ω

similar arguments as those in the proof of Lemma 2.7.4. Note that the limiting objective function given by (3.67) is an affine transformation of the logarithmic utility $U(z) = \log z, z > 0$ for $c = 1$ or the power utility $U(z) = \frac{z^{1-c}}{1-c}, z > 0$, for $c \in (0,1) \cup (1,+\infty)$; and the corresponding optimizers are invariant w.r.t. such transformations. As a result, Theorem 2.7.3 entails that $(\widehat{X}_T^m)_{m\in\mathbb{N}}$ (resp. $(\widehat{Y}_T^m)_{m\in\mathbb{N}}$) converges in probability to the optimal terminal wealth (resp. terminal value of the dual optimizer) for the utility maximization problem with objective function given by the logarithmic utility for $c = 1$ and power utility for $c \in (0,1) \cup (1,+\infty)$.

We recall some continuity results on which we will rely for our study of risk aversion asymptotics. We begin with the following stability results for BSDEs.

Theorem 3.5.6. *[MW12, Theorem 7] Suppose that Assumption 3.4.7 holds and the filtration $\mathbb{F}$ is continuous. Let F be a convex driver satisfying the conditions i), ii) and iii) of Theorem 3.4.8 and (Y, Z, N) the unique solution to the BSDE$(F, 0)$ in the space $\Xi \times \mathcal{H}^2 \times \mathcal{M}^2$. Let $(F^m)_{m\in\mathbb{N}}$ be a sequence of drivers such that for each m, F^m is convex in z and satisfies the conditions i), ii) and iii) of Theorem 3.4.8. Let (Y^m, Z^m, N^m) be the solution in $\Xi \times \mathcal{H}^2 \times \mathcal{M}^2$ associated to $(F^m, 0)$. Suppose additionally that as m goes to $+\infty$,*

$$\int_0^T |F^m(t, Z_t) - F(t, Z_t)| dK_t \to 0 \text{ in probability.}$$

Then for each $l \geq 1$ and as m goes to $+\infty$,

$$\mathbb{E}\left[\left(\exp\left(\sup_{t\in[0,T]} |Y_t - Y_t^m|\right)\right)^l\right] \to 1 \text{ and } ||Z^m - Z||_{\mathcal{H}^l} + ||N^m - N||_{\mathcal{M}^l} \to 0.$$

Let us recall the following continuity result for stochastic exponential of BMO martingales.

Theorem 3.5.7. *[Kaz94, Theorem 3.2] Let $\mathbb{Q}$ be a probability measure equivalent to $\mathbb{P}$. We denote by $\mathbb{H}_1(\mathbb{Q})$ the Hardy space of continuous $\mathbb{Q}$-local martingales N equipped with the norm $||N||_{\mathbb{H}_1(\mathbb{Q})} = \mathbb{E}^{\mathbb{Q}}[\langle N\rangle_T^{\frac{1}{2}}]$. Then the mapping $BMO(\mathbb{Q}) \ni N \mapsto \mathcal{E}(N) - 1 \in \mathbb{H}_1(\mathbb{Q})$ is continuous.*

3.5.1 The limit $c \in (0,1) \cup (1,+\infty)$

The sequences $(\Phi_{U_m})_{m\in\mathbb{N}}$ and $(A_{U_m})_{m\in\mathbb{N}}$ converge uniformly to $\phi_{1-c} = \frac{1}{2}\frac{c-1}{c}$ and c respectively, and the corresponding limits are identical to Φ_U and A_U with $U(z) = \frac{z^{1-c}}{1-c}, z > 0$. We consider the power utility maximization problem with relative risk aversion c, i.e.

$$\overline{u}_{1-c}(x_0) := \sup_{\pi\in\mathcal{A}(x_0)} \mathbb{E}\left[U(X_T^\pi)\right]. \tag{3.75}$$

Assuming that [7] $\overline{u}_{1-c} < +\infty$, let $L(1-c)$ be the opportunity process for the power utility function $U(z) = \frac{z^{1-c}}{1-c}, z > 0$. By Theorem 3.3.5, there exists $Z^{L(1-c)} \in \mathcal{L}(M)$ and $N^{L(1-c)}$ a local martingale orthogonal to M such that for $t \in [0,T]$

$$dL_t(1-c) \quad = \quad Z_t^{L(1-c)} dM_t + dN_t^{L(1-c)} + L_t(1-c)\phi_{1-c}\left(\mu_t + \xi_t^{L(1-c)}\right)^\top d\langle M\rangle_t \left(\mu_t + \xi_t^{L(1-c)}\right) \tag{3.76}$$

where

$$\xi^{L(1-c)} = \frac{Z^{L(1-c)}}{L(1-c)}, \quad \Pi^{L(1-c)} = \int_0^\cdot \frac{1}{L(1-c)} dN^{L(1-c)} \text{ and } \phi_{1-c} = \frac{1}{2}\frac{c-1}{c}. \tag{3.77}$$

[7] This is always guaranteed for $c \in (1,+\infty)$. For $c \in (0,1)$, it is guaranteed as soon as $\mathbb{E}\left[\left(Z_T^\mu\right)^{\frac{c-1}{c}}\right]$, see Remark 2.3.1. A sufficient condition for the latter is Assumption 3.4.7 or $b(\mu \cdot M) > \kappa(\frac{1-c}{c})$.

The optimal trading strategy, wealth process and dual optimizers for the problem (3.75) are given respectively by $\nu^{L(1-c)}$, $x_0\widehat{X}(1-c)$ and $\overline{u}'_{1-c}(x_0)\widehat{Y}(1-c)$ where

$$\nu^{L(1-c)} = \frac{1}{c}(\mu + \xi^{L(1-c)}), \widehat{X}(1-c) = \mathcal{E}\left(\nu^{L(1-c)} \cdot R\right) \text{ and } \widehat{Y}(1-c) = \mathcal{E}\left(-\mu \cdot M + \Pi^{L(1-c)}\right). \tag{3.78}$$

The following theorem gives the convergence of the optimizers under the exponential moments condition on $\int_0^T ||\sigma_s \mu_s||^2 dK_s$. We recall that $\mathcal{S}_0$ is the semimartingale topology.

Theorem 3.5.8. *Suppose that Assumption 3.4.7 holds. Let $(U_m)_{m\in\mathbb{N}}$ be a sequence of utility functions satisfying Assumption 3.5.1. Let $(\nu^m)_{m\in\mathbb{N}}, (\widehat{X}^m)_{m\in\mathbb{N}}$ and $(\widehat{Y}^m)_{m\in\mathbb{N}}$ be the sequence of optimizers given by (3.71) and $\nu^{L(1-c)}, \widehat{X}(1-c)$ and $\widehat{Y}(1-c)$ be defined by (3.78). Then as $m \to +\infty$:*

A1. $||\nu^m - \nu^{L(1-c)}||_{\mathcal{H}^2} \to 0,$

A2. $\widehat{X}^m \to x_0\widehat{X}(1-c)$ *in* $\mathcal{S}_0$ *and* $\widehat{Y}^m/\widehat{Y}_0^m \to \widehat{Y}(1-c)$ *in* $\mathcal{S}_0$.

The proof relies on the following lemma.

Lemma 3.5.9. *Under the assumptions and notation of Theorem 3.5.8, as $m \to +\infty$:*

$$||\xi^{L^m} - \xi^{L(1-c)}||_{\mathcal{H}^2} + ||\Pi^{L^m} - \Pi^{L(1-c)}||_{\mathcal{M}^2} \to 0. \tag{3.79}$$

Proof. First we give some properties. Let $m \in \mathbb{N}$. Due to H2), H3) and Assumption 3.4.7, Theorem 3.4.10 and Proposition 3.4.14 entail that $\left(\log L^m, \xi^{L^m}, \Pi^{L^m}\right) \in \varXi \times \mathcal{H}^2 \times \mathcal{M}^2$. As $F_{U_m}(\cdot, \widehat{X}^m, \cdot)$ is convex, locally Lipschitz, and has quadratic growth in z (see Remark 3.4.5), we infer from Theorem 3.4.8 that $\left(\log L^m, \xi^{L^m}, \Pi^{L^m}\right)$ is the unique solution to the BSDE $(F_{U_m}(\cdot, \widehat{X}^m, \cdot), 0)$ in the space $\varXi \times \mathcal{H}^2 \times \mathcal{M}^2$. One verifies that $\left(\log L(1-c), \xi^{L(1-c)}, \Pi^{L(1-c)}\right)$ is a solution to the BSDE$(f_{1-c}, 0)$, and f_{1-c} is as well locally Lipschitz, convex and has quadratic growth in z. Since $\log L(1-c) \in \varXi$ by Lemma A.6 in [MW13], Theorem 3.4.8 ensures that $\left(\log L(1-c), \xi^{L(1-c)}, \Pi^{L(1-c)}\right) \in \varXi \times \mathcal{H}^2 \times \mathcal{M}^2$. Moreover, it is the unique solution to the BSDE$(f_{1-c}, 0)$ in the aforementioned space.
Now, (3.79) follows from Theorem 3.5.6 as soon as we show that

$$\lim_{m\to+\infty} \int_0^T |F_{U_m}(t, \widehat{X}_t^m, \xi_t^{L(1-c)}) - f_{1-c}(t, \xi_t^{L(1-c)})| dK_t = 0 \text{ in probability.} \tag{3.80}$$

Let $m \in \mathbb{N}$, using the expressions of F_{U_m} and f_{1-c} given respectively by (3.70) and (3.72), we have for $t \in [0, T]$

$$|F_{U_m}(t, \widehat{X}_t^m, \xi_t^{L(1-c)}) - f_{1-c}(t, \xi_t^{L(1-c)})| \quad = \quad |\Phi_{U_m}(\widehat{X}_t^m) - \phi_{1-c}| \times ||\sigma_t(\mu_t + \xi_t^{L(1-c)})||^2.$$

Hence with $\rho_m = ||\Phi_{U_m}(\widehat{X}_m) - \phi_{1-c}||_\infty$, the above estimate implies that

$$\begin{aligned} \mathbb{E}\left[\int_0^T |F_{U_m}(t, \widehat{X}_t^m, \xi_t^{L(1-c)}) - f_{1-c}(t, \xi_t^{L(1-c)})| dK_t\right] &\le \rho_m \mathbb{E}\left[\int_0^T ||\sigma_t(\mu_t + \xi_t^{L(1-c)})||^2 dK_t\right] \\ &\le \rho_m ||\mu + \xi^{L(1-c)}||^2_{\mathcal{H}^2}. \end{aligned}$$

By hypothesis, $(\Phi_{U_m})_{m\in\mathbb{N}}$ converges to ϕ_{1-c} uniformly. Thus $\lim_{m\to+\infty} \rho_m = 0$. Consequently, (3.80) holds and the proof is complete. $\square$

Proof of Theorem 3.5.8. A1. For $m \in \mathbb{N}$, set $\gamma^m := \frac{1}{A_{U_m}(\widehat{X}^m)} - \frac{1}{c}$ and $\delta^m := \frac{1}{A_{U_m}(\widehat{X}^m)}$. We recall that for $m \in \mathbb{N}$, $\nu^m = \frac{1}{A_{U_m}(\widehat{X}^m)}(\mu + \xi^{L^m})$ and $\nu^{L(1-c)} = \frac{1}{c}(\mu + \xi^{L(1-c)})$. Clearly for $m \in \mathbb{N}$, we have

$$\nu^m - \nu^{L(1-c)} = \gamma^m(\mu + \xi^{L(1-c)}) + \delta^m(\xi^{L^m} - \xi^{L(1-c)}). \tag{3.81}$$

An application of the binomial inequalities yields for $m \in \mathbb{N}$

$$\begin{aligned} ||\nu^m - \nu^{L(1-c)}||^2_{\mathcal{H}^2} &= \mathbb{E}\left[\int_0^T ||\sigma_s(\nu^m_s - \nu^{L(1-c)}_s)||^2 dK_s\right] \\ &= \mathbb{E}\left[\int_0^T ||\sigma_s\gamma^m_s(\mu_s + \xi^{L(1-c)}_s) + \delta^m_s\sigma_s(\xi^{L^m}_s - \xi^{L(1-c)}_s)||^2 dK_s\right] \\ &\leq 2||\gamma^m||^2_\infty \times ||\mu + \xi^{L(1-c)}||^2_{\mathcal{H}^2} + 2||\delta^m||^2_\infty \times ||\xi^{L^m} - \xi^{L(1-c)}||^2_{\mathcal{H}^2}. \end{aligned}$$

As $(A_{U_m})_{m\in\mathbb{N}}$ converges uniformly to c, $\lim_{m\to+\infty} ||\gamma^m||_\infty = 0$ and $\lim_{m\to+\infty} ||\delta^m||_\infty = 1/c$. We infer from Lemma 3.5.9 and the above estimate that $\lim_{m\to+\infty} ||\nu^m - \nu^{L(1-c)}||^2_{\mathcal{H}^2} = 0$.
A2. We begin with the convergence of the optimal wealth processes. We recall that $dR = dM + d\langle M\rangle\mu$, $\widehat{X}(1-c) = \mathcal{E}(\nu^{L(1-c)} \cdot R)$ and for $m \in \mathbb{N}$, $\widehat{X}^m = x_0\mathcal{E}(\nu^m \cdot R)$. Under Assumption 3.4.7, $Z^\mu = \mathcal{E}(-\mu \cdot M)$ is a true martingale and $\mathbb{E}\left[(Z^\mu_T)^2\right] < +\infty$ by Proposition 3.4.9. Let $\mathbb{Q}^\mu$ be the probability measure defined by $d\mathbb{Q}^\mu/d\mathbb{P} = Z^\mu = \mathcal{E}(-\mu \cdot M)$. As $\mathbb{P}$ and $\mathbb{Q}^\mu$ are equivalent, the topology $\mathcal{S}_0$ remains unchanged if we replace $\mathbb{P}$ by $\mathbb{Q}^\mu$ in the Emery distance (2.3), see [Eme79, Proposition 6]. Using Bayes' formula and Hölder's inequalities, we have for $m \in \mathbb{N}$

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}^\mu}\left[\left[\int_0^\cdot \left(\nu^m - \nu^{L(1-c)}\right) dR\right]^{\frac{1}{2}}_T\right] &= \mathbb{E}^{\mathbb{Q}^\mu}\left[\left(\int_0^T ||\sigma_s(\nu^m_s - \nu^{L(1-c)}_s)||^2 dK_s\right)^{\frac{1}{2}}\right] \\ &\leq \left(\mathbb{E}\left[(Z^\mu_T)^2\right]\right)^{\frac{1}{2}} \times \left(\mathbb{E}\left[\int_0^T ||\sigma_s(\nu^m_s - \nu^{L(1-c)}_s)||^2 dK_s\right]\right)^{\frac{1}{2}} \\ &\leq \left(\mathbb{E}\left[(Z^\mu_T)^2\right]\right)^{\frac{1}{2}} \times ||\nu^m - \nu^{L(1-c)}||_{\mathcal{H}^2}. \end{aligned}$$

Assertion A1. and the above estimate give

$$\lim_{m\to+\infty} \mathbb{E}^{\mathbb{Q}^\mu}\left[\left[\int_0^\cdot \left(\nu^m - \nu^{L(1-c)}\right) dR\right]^{\frac{1}{2}}_T\right] = 0.$$

Since by Girsanov's theorem, $\nu^{L(1-c)} \cdot R$ and $\nu^m \cdot R$ are $\mathbb{Q}^\mu$-local continuous martingales for each $m \in \mathbb{N}$, it follows from Proposition 2.2.9 that as $m \to +\infty$, we have $\nu^m \cdot R \to \nu^{L(1-c)} \cdot R$ in $\mathcal{S}_0$ and $\widehat{X}^m/x_0 = \mathcal{E}(\nu^m \cdot R) \to \widehat{X}(1-c) = \mathcal{E}(\nu^{L(1-c)} \cdot R)$ in $\mathcal{S}_0$,
For the convergence of dual optimizers, we argue similarly. As $\left(\Pi^{L^m} - \Pi^{L(1-c)}\right)_{m\in\mathbb{N}}$ is a sequence of martingales, using once more Proposition 2.2.9, Lemma 3.5.9 implies that

$$\lim_{m\to+\infty} \Pi^{L^m} = \Pi^{L(1-c)} \text{ in } \mathcal{S}_0.$$

Now $\widehat{Y}^m/\widehat{Y}_0 = \mathcal{E}(-\mu \cdot M + \Pi^{L^m})$ for $m \in \mathbb{N}$ and $\widehat{Y}(1-c) = \mathcal{E}(-\mu \cdot M + \Pi^{L(1-c)})$. As $(\Pi^{L^m})_{m\in\mathbb{N}}$ is a sequence of continuous local martingales converging in $\mathcal{S}_0$, Proposition 2.2.9 guarantees that $(\widehat{Y}^m/\widehat{Y}_0)_{m\in\mathbb{N}}$ converge to $\widehat{Y}(1-c)$ in $\mathcal{S}_0$. □

Remark 3.5.10. *Theorem 3.5.8 generalizes Theorem 3.8 in [MW13] for which $(U_m)_{m\in\mathbb{N}}$ has the form E1) in Example 3.5.3. Note however, that in [MW13], trading strategies are subject to constraints and the underlying market price of risk μ is allowed to vary.*

The following theorem gives the convergence of the sequence of the optimizers under a BMO condition on $\mu \cdot M$. In comparison to the general stability result on utility maximization given by Theorem 2.7.5, Assumption 3.5.1 enables us to obtain the convergence of the opportunity processes and optimal trading strategies in stronger topologies. In addition, we also have convergence of the dual optimizers in entropy.

Theorem 3.5.11. *We keep the notation of Theorem 3.5.8. Let κ be defined by (3.30), i.e.*

$$\kappa(z) = z^2 + +\frac{1}{2}z + z\sqrt{z(z+1)}, z > 0. \tag{3.82}$$

We consider the following two cases:

i) $c \in (0,1)$ *and there exists* $\epsilon > 0$ *such that* $b\,(\mu \cdot M) > \kappa(\frac{1-c}{c} + 2\epsilon)$.

ii) $c \in (1,+\infty)$ *and* $\mu \cdot M \in BMO$.

Set $d\widehat{\mathbb{Q}}(1-c)/d\mathbb{P} := \widehat{Y}_T(1-c)$ *and* $d\widehat{\mathbb{Q}}^m/d\mathbb{P} = \widehat{Y}_T^m/\widehat{Y}_0^m$ *for* $m \in \mathbb{N}$. *If either i) or ii) holds, then as* $m \to \infty$,

a) $||\log L^m - \log L(1-c)||_\infty \to 0$,

b) $||(\nu^m - \nu^{L(1-c)}) \cdot M||_{BMO} \to 0$,

c) $||\widehat{X}^m - x_0\widehat{X}(1-c)||_{\mathbb{H}_1(\mathbb{Q}^\mu)} \to 0$,

d) $||\widehat{Y}^m/\widehat{Y}_0^m - \widehat{Y}(1-c)||_{\mathbb{H}_1} + \mathcal{I}\left(\widehat{\mathbb{Q}}^m, \widehat{\mathbb{Q}}(1-c)\right) \to 0$.

Remark 3.5.12. *The key assertion in Theorem 3.5.11 is the convergence of the sequence of opportunity processes* $(L^m)_{m\in\mathbb{N}}$ *in the norm* $||\cdot||_\infty$. *All subsequent assertions will be based on this convergence. Observe that the convergence of* $(\log L^m)_{m\in\mathbb{N}}$ *given by assertion a) generalizes the convergence at each date given by Theorem 2.7.5. This is also the case for the sequence* $(\nu^m \cdot M)_{m\in\mathbb{N}}$. *Recall however that in Theorem 2.7.5 we required less differentiability and the filtration need not be continuous.*

Proof of Theorem 3.5.11. a) We consider the case $c \in (0,1)$. For the convergence of the sequence $(\log L^m)_{m\in\mathbb{N}}$, we will adapt accordingly some arguments from [MT03b, Lemma 4.2] showing the Lipschitz property of the opportunity process w.r.t. risk aversion in the particular case of power utilities. First we show that the sequence $\left(\log L^m - \log L(1-c)\right)_{m\in\mathbb{N}}$ is uniformly bounded from a certain rank. The function κ being increasing, we can assume that $\epsilon < \frac{1}{2c}$. As $(\Phi_{U_m})_{m\in\mathbb{N}}$ converges to $\frac{1}{2}\frac{c-1}{c}$, there exists $N_\epsilon > 0$ such that for all $m \geq N_\epsilon$,

$$\underline{\alpha}_\epsilon = \frac{1}{2}\frac{c-1}{c} - \epsilon \leq \Phi_{U_m} \leq \frac{1}{2}\frac{c-1}{c} + \epsilon = \overline{\alpha}_\epsilon.$$

Following Remark 3.4.4, the above inequality together with hypotheses H2) and H3) of Assumption 3.5.1 imply that for $m \geq N_\epsilon$, U_m satisfies $(G_{a_\epsilon,b_\epsilon,1})$ where $a_\epsilon = \frac{1}{1-2\underline{\alpha}_\epsilon}$ and $b_\epsilon = \frac{1}{1-2\overline{\alpha}_\epsilon}$. Since $a_\epsilon \in (0,1)$ and $b\,(\mu \cdot M) > \kappa(\frac{1-c}{c} + 2\epsilon) = \kappa(\frac{1-a_\epsilon}{a_\epsilon})$, by Proposition 3.4.21 there exists $K_\infty > 0$ such that

$$\sup_{m \geq N_\epsilon} ||\log L^m||_\infty + ||\log L(1-c)||_\infty \leq K_\infty. \tag{3.83}$$

Let $m \geq N_\epsilon$. We recall the following equation describing the dynamics of $\log L^m$ and $\log L(1-c)$. For $t \in [0,T]$

$$\begin{aligned} d\log L_t^m &= \xi_t^{L^m} dM_t + d\Pi_t^{L^m} - F_{U_m}(t, \widehat{X}_t^m, \xi_t^{L^m})dK_t - \frac{1}{2}d\left\langle \Pi^{L^m}\right\rangle_t, \\ d\log L_t(1-c) &= \xi_t^{L(1-c)} dM_t + d\Pi_t^{L(1-c)} - f_{1-c}(t, \xi_t^{L(1-c)})dK_t - \frac{1}{2}d\left\langle \Pi^{L(1-c)}\right\rangle_t. \end{aligned}$$

We make the following change of variables:

$$\begin{cases} \psi^m & = & \log L^m - \log L(1-c) \\ \xi^m & = & \xi^{L^m} - \xi^{L(1-c)} \\ \Pi^m & = & \Pi^{L^m} - \Pi^{L(1-c)}, \\ \Delta^m F & = & F_{U_m}(\cdot, \widehat{X}^m, \xi^{L^m}) - f_{1-c}(\cdot, \xi^{L(1-c)}). \end{cases}$$

Let $\tau \in \mathcal{T}(\mathbb{F})$. From the identity $\left\langle \Pi^{L^m} \right\rangle - \left\langle \Pi^{L(1-c)} \right\rangle = \left\langle \Pi^m, \Pi^{L^m} + \Pi^{L(1-c)} \right\rangle$,the definition of ψ^m and $\psi^m_T = 0$, we have

$$-\psi^m_\tau \;=\; \int_\tau^T \xi^m dM_t + \Pi^m_T - \Pi^m_\tau - \frac{1}{2}\int_\tau^T d\left[\Pi^m, \Pi^{L^m} + \Pi^{L(1-c)}\right]_t - \int_\tau^T \Delta^m F(t) dK_t.$$

Now for $t \in [0,T]$, we can write $-\Delta^m F(t)$ as follows

$$\begin{aligned} -\Delta^m F(t) &= \Phi_{U_m}(\widehat{X}^m_t)||\sigma_t(\mu_t + \xi^{L^m}_t)||^2 - \phi_{1-c}||\sigma_t(\mu_t + \xi^{L(1-c)}_t)||^2 \\ &\quad + \frac{1}{2}||\sigma_t\xi^{L(1-c)}_t||^2 - \frac{1}{2}||\sigma_t\xi^{L^m}_t||^2 \\ &= (\Phi_{U_m}(\widehat{X}^m_t) - \phi_{1-c})||\sigma_t(\mu_t + \xi^{L^m}_t)||^2 - \frac{1}{2}(\sigma_t\xi^m_t)^\top(\sigma_t\xi^{L(1-c)}_t + \sigma_t\xi^{L^m}_t) \\ &\quad + (\sigma_t\xi^m_t)^\top(2\phi_{1-c}\sigma_t\mu_t + \phi_{1-c}\sigma_t\xi^{L^m}_t + \phi_{1-c}\sigma_t\xi^{L(1-c)}_t) \\ &= (\Phi_{U_m}(\widehat{X}^m_t) - \phi_{1-c})||\sigma_t(\mu_t + \xi^{L^m}_t)||^2 \\ &\quad + (\sigma_t\xi^m_t)^\top\left(2\phi_{1-c}\sigma_t\mu_t + (\phi_{1-c} - \frac{1}{2})\sigma_t\xi^{L^m}_t + (\phi_{1-c} - \frac{1}{2})\sigma_t\xi^{L(1-c)}_t\right). \end{aligned}$$

Let $\zeta^m, N^m, \widehat{M}^m$ and $\widehat{\Pi}^m$ be the processes defined by:

$$\begin{aligned} \zeta^m &= -2\phi_{1-c}\mu - (\phi_{1-c} - \frac{1}{2})\xi^{L^m} - (\phi_{1-c} - \frac{1}{2})\xi^{L(1-c)} \\ N^m &= \int_0^\cdot \zeta^m dM + \frac{1}{2}(\Pi^{L^m} + \Pi^{L(1-c)}), \\ \widehat{M}^m &= \int_0^\cdot \xi^m(dM + d\langle M\rangle\zeta^m), \text{ and } \widehat{\Pi}^m = \Pi^m - \frac{1}{2}\left\langle \Pi^m, \Pi^{L(1-c)} + \Pi^{L^m}\right\rangle. \end{aligned} \tag{3.84}$$

With the above decomposition of $-\Delta^m F$, $\zeta^m, \widehat{M}^m$ and $\widehat{\Pi}^m$, $-\psi^m_\tau$ takes the form

$$-\psi^m_\tau = \int_\tau^T \xi^m_t(dM_t + d\langle M\rangle_t\zeta^m_t) + \Pi^m_T - \Pi^m_\tau - \frac{1}{2}\int_\tau^T d\left\langle \Pi^m, \Pi^{L^m} + \Pi^{L(1-c)}\right\rangle_t \tag{3.85}$$

$$+ \int_\tau^T (\Phi_{U_m}(\widehat{X}^m_t) - \phi_{1-c})||\sigma_t(\mu_t + \xi^{L^m}_t)||^2 dK_t \tag{3.86}$$

$$= \widehat{M}^m_T - \widehat{M}^m_\tau + \widehat{\Pi}^m_T - \widehat{\Pi}^m_\tau + \int_\tau^T \left(\Phi_{U_m}(\widehat{X}^m_t) - \phi_{1-c}\right)||\sigma_t(\mu_t + \xi^{L^m}_t)||^2 dK_t. \tag{3.87}$$

Since $\mu \cdot M, \int_0^\cdot \xi^{L^m} dM, \int_0^\cdot \xi^{L(1-c)} dM, \Pi^{L^m}$ and $\Pi^{L(1-c)}$ are BMO martingales, N^m is a BMO martingale. Furthermore, as Π^{L^m} and $\Pi^{L(1-c)}$ are orthogonal to M

$$\widehat{M}^m \;=\; \int_0^\cdot \xi^m dM - \left\langle \int_0^\cdot \xi^m dM, N^m \right\rangle \text{ and } \widehat{\Pi}^m = \Pi^m - \langle \Pi^m, N^m\rangle.$$

Employing Theorem 3.2.6, $\widehat{M}^m, \widehat{\Pi}^m \in BMO(\mathbb{Q}^m)$ where $\mathbb{Q}^m$ is the probability measure equivalent to $\mathbb{P}$ with density given by

$$d\mathbb{Q}^m/d\mathbb{P} \;=\; \widehat{Z}^m_T \text{ where } \widehat{Z}^m = \mathcal{E}(N^m).$$

Taking conditional expectation in (3.87) w.r.t. the measure $\mathbb{Q}^m$ gives

$$-\psi_\tau^m = \mathbb{E}^{\mathbb{Q}^m}\left[\int_\tau^T (\Phi_{U_m}(\widehat{X}_t^m) - \phi_{1-c})||\sigma_t(\mu_t + \xi_t^{L^m})||^2 dK_t \Big| \mathcal{F}_\tau\right].$$

The above equality implies that

$$|\psi_\tau^m| \leq \rho_m \mathbb{E}^{\mathbb{Q}^m}\left[\int_\tau^T ||\sigma_t(\mu_t + \xi_t^{L^m})||^2 dK_t \Big| \mathcal{F}_\tau\right], \tag{3.88}$$

where

$$\rho_m = ||\Phi_{U_m}(\widehat{X}^m) - \phi_{1-c}||_\infty. \tag{3.89}$$

By hypothesis ii), $(\rho_m)_{m\in\mathbb{N}}$ converges to 0 as m goes to $+\infty$. Therefore to show a), it remains to show that the other multiplicative factor in the right hand of (3.88) is uniformly bounded for $m \geq N_\epsilon$ and $\tau \in \mathcal{T}(\mathbb{F})$. To this end, we apply Theorem 3.4.23 and use the uniform bound (3.83) to guarantee the existence of a constant K_{BMO} depending only on $||\mu \cdot M||_{BMO}$ such that for all $m \geq N_\epsilon$,

$$\left|\left|\int_0^\cdot \xi^{L^m} dM\right|\right|_{BMO}^2 + \left|\left|\int_0^\cdot \xi^{L(1-c)} dM\right|\right|_{BMO}^2 + \left|\left|\Pi^{L^m}\right|\right|_{BMO}^2 + \left|\left|\Pi^{L(1-c)}\right|\right|_{BMO}^2 \leq K_{BMO}. \tag{3.90}$$

We deduce from the expression (3.84) of N^m and the above bound that there exists $\delta > 0$ which depends only on c and $||\mu \cdot M||_{BMO}$ such that for all $m \geq N_\epsilon$, we have

$$||N^m||_{BMO} \leq \delta.$$

Recalling that for $m \in \mathbb{N}$, we have $\widehat{Z}^m = \mathcal{E}(N^m)$ the above bound and Theorem 3.2.6 imply that there exists $k > 1$ such that

$$\sup_{m\geq N_\epsilon,\ \tau\in\mathcal{T}(\mathbb{F})} \mathbb{E}\left[\left(\frac{\widehat{Z}_T^m}{\widehat{Z}_\tau^m}\right)^k \Big| \mathcal{F}_\tau\right] \leq \eta,$$

where η is a constant depending only on δ. Let $r \in \mathbb{N}$ such that $\frac{r}{r-1} < k$,$m \geq N_\epsilon$ and $\tau \in \mathcal{T}(\mathbb{F})$. Then Hölder's inequality yields

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^m}\left[\int_\tau^T ||\sigma_t(\mu_t + \xi_t^{L^m})||^2 dK_t \Big| \mathcal{F}_\tau\right] &= \mathbb{E}\left[\frac{\widehat{Z}_T^m}{\widehat{Z}_\tau^m}\int_\tau^T ||\sigma_t(\mu_t + \xi_t^{L^m})||^2 dK_t \Big| \mathcal{F}_\tau\right] \\
&\leq \left(\mathbb{E}\left[\left(\frac{\widehat{Z}_T^m}{\widehat{Z}_\tau^m}\right)^{\frac{r}{r-1}} \Big| \mathcal{F}_\tau\right]\right)^{\frac{r-1}{r}} \\
&\quad \times \left(\mathbb{E}\left[\left(\int_\tau^T ||\sigma_t(\mu_t + \xi_t^{L^m})||^2 dK_t\right)^r \Big| \mathcal{F}_\tau\right]\right)^{\frac{1}{r}}.
\end{aligned}$$

The first term on the right hand of the above inequality is uniformly bounded. Indeed, Hölder's inequality and the bound η give

$$\left(\mathbb{E}\left[\left(\frac{\widehat{Z}_T^m}{\widehat{Z}_\tau^m}\right)^{\frac{r}{r-1}} \Big| \mathcal{F}_\tau\right]\right)^{\frac{r-1}{r}} \leq \left(\mathbb{E}\left[\left(\frac{\widehat{Z}_T^m}{\widehat{Z}_\tau^m}\right)^{k} \Big| \mathcal{F}_\tau\right]\right)^{\frac{1}{k}} \leq \eta^{\frac{1}{k}}.$$

The second term on the right hand of the previous inequality is also uniformly bounded. This follows from the application of the energy inequalities (see [DM82b, Remarks 106.2]) from which we obtain

$$\left(\mathbb{E}\left[\left(\int_\tau^T ||\sigma_t(\mu_t+\xi_t^{L^m})||^2 dK_t\right)^r \Bigg| \mathcal{F}_\tau\right]\right)^{\frac{1}{r}} \leq \left(r!\left|\left|\int_0^\cdot \left(\mu+\xi^{L^m}\right)dM\right|\right|_{BMO}^{2r}\right)^{\frac{1}{r}} \leq K'_{BMO}, \tag{3.91}$$

where K'_{BMO} is a constant depending on $||\mu\cdot M||_{BMO}$ and c. We infer from (3.88) that

$$||\psi^m||_\infty \leq \rho_m \eta^{\frac{1}{k}} K'_{BMO} \to 0 \text{ as } m\to+\infty.$$

b) We now prove the second statement. First we show that as $m\to\infty$,

$$\left|\left|\int_0^\cdot \xi^m dM\right|\right|_{BMO} + ||\Pi^m||_{BMO} \to 0. \tag{3.92}$$

We will apply standard arguments for BSDEs to obtain a control of the norm in (3.92). Let Ψ be the function given by

$$\Psi : \mathbb{R}\ni z \to e^z - z - 1.$$

Let $m\in\mathbb{N}$ and $\tau\in\mathcal{T}(\mathbb{F})$. As $\psi_T^m = 0$ and $\Psi(0)=0$, Itô's formula and (3.85) imply that

$$\begin{aligned}
\Psi(\psi_\tau^m) + \frac{1}{2}\int_\tau^T \Psi''(\psi_s^m)d\left[\psi^m\right]_s &= -\int_\tau^T \Psi'(\psi_s^m)d\psi_s^m \\
&= -\int_\tau^T \Psi'(\psi_s^m)\xi_s^m\left(dM_s + d\langle M\rangle_s\zeta_s^m\right) - \int_\tau^T \Psi'(\psi_s^m)d\Pi_s^m \\
&\quad + \frac{1}{2}\int_\tau^T \Psi'(\psi_s^m)d\left\langle \Pi^m, \Pi^{L(1-c)}+\Pi^{L^m}\right\rangle_s \\
&\quad - \int_\tau^T \Psi'(\psi_s^m)(\Phi_{U_m}(\widehat{X}_s^m) - \phi_{1-c})||\sigma_s(\mu_s+\xi_s^{L^m})||^2 dK_s.
\end{aligned}$$

Since ψ^m is bounded, $\Psi'(\psi^m)$ is bounded. Thus the BMO properties of $\int_0^\cdot \xi^m dM$ and Π^m entail that the local martingales $\int_0^\cdot \Psi'(\psi^m)\xi^m dM$ and $\int_0^\cdot \Psi'(\psi^m)d\Pi^m$ are true martingales. Therefore taking conditional expectations in the above equation and using the fact that $\Psi\geq 0$ give

$$\begin{aligned}
\frac{1}{2}\mathbb{E}\left[\int_\tau^T \Psi''(\psi_s^m)d\left[\psi^m\right]_s\Big|\mathcal{F}_\tau\right] &\leq -\mathbb{E}\left[\int_\tau^T \Psi'(\psi_s^m)\left(\sigma_s\xi_s^m\right)^\top(\sigma_s\zeta_s^m)dK_s\Big|\mathcal{F}_\tau\right] \\
&\quad + \frac{1}{2}\mathbb{E}\left[\int_\tau^T \Psi'(\psi_s^m)d\left\langle \Pi^m,\Pi^{L(1-c)}+\Pi^{L^m}\right\rangle_s\Big|\mathcal{F}_\tau\right] \\
&\quad - \mathbb{E}\left[\int_\tau^T \Psi'(\psi_s^m)(\Phi_{U_m}(\widehat{X}_s^m)-\phi_{1-c})||\sigma_s(\mu_s+\xi_s^{L^m})||^2 dK_s\Big|\mathcal{F}_\tau\right].
\end{aligned} \tag{3.93}$$

We will now give an appropriate bound for each term on the right hand side of the above inequality. Using the binomial inequalities $|ab|\leq\frac{1}{2}(a^2+b^2)$ and the inequality of Kunita-Watanabe, we have

$$\left|\mathbb{E}\left[\int_\tau^T \Psi'(\psi_s^m)\left(\sigma_s\xi_s^m\right)^\top(\sigma_s\zeta_s^m)dK_s\Big|\mathcal{F}_\tau\right]\right| \leq \frac{1}{2}||\Psi'(\psi^m)||_\infty\left(\left|\left|\int_0^\cdot \xi^m dM\right|\right|_{BMO}^2 + \left|\left|\int_0^\cdot \zeta^m dM\right|\right|_{BMO}^2\right),$$

and

$$\left|\mathbb{E}\left[\int_\tau^T \Psi'(\psi_s^m)d\left[\Pi^m,\Pi^{L(1-c)}+\Pi^{L^m}\right]_s\Big|\mathcal{F}_\tau\right]\right| \leq \frac{1}{2}||\Psi'(\psi^m)||_\infty\left(||\Pi^m||_{BMO}^2 + ||\Pi^{L^m}+\Pi^{L(1-c)}||_{BMO}^2\right).$$

With $\rho_m = ||\Phi_{U_m}(\widehat{X}^m) - \phi_{1-c}||_\infty$, we have the following estimate for the last term

$$\left|\mathbb{E}\left[\int_\tau^T \Psi^{'}(\psi_s^m)(\Phi_{U_m}(\widehat{X}_t^m) - \phi_{1-c})||\sigma_t(\mu_t + \xi_t^{L^m})||^2 dK_t \Big| \mathcal{F}_\tau\right]\right|$$
$$\leq \rho_m ||\Psi^{'}(\psi^m)||_\infty \left|\left|\int_0^\cdot \left(\mu + \xi^{L^m}\right) dM\right|\right|^2_{BMO}.$$

We recall that ξ^m is given by (3.84). Putting the above estimate together and using the uniform bound (3.90) with additional simple inequalities, we deduce from (3.93) that

$$\frac{1}{2}\mathbb{E}\left[\int_\tau^T \Psi^{''}(\psi_s^m) d\left[\psi^m\right]_s \Big| \mathcal{F}_\tau\right] \leq ||\Psi^{'}(\psi^m)||_\infty \left(k_m^{'}\, ||\mu\cdot M||^2_{BMO} + k_m^{''} K_{BMO}\right), \quad (3.94)$$

where $k_m^{'}$ and $k_m^{''}$ are given by

$$k_m^{'} = 2\rho_m + 6|\phi_{1-c}|^2 \text{ and } k_m^{''} = 2\rho_m + 3 + 3|\phi_{1-c}|^2.$$

Now $\Psi^{''}(\psi^m) = e^{\psi^m}$ and for every $s \in [0,T], e^{-||\psi^m||_\infty} \leq e^{\psi_s^m}$. The function $\Psi^{'}$ being increasing, $||\Psi^{'}(\psi^m)||_\infty \leq \Psi^{'}(||\psi^m||_\infty)$. Clearly $[\psi^m] = \int_0^\cdot ||\sigma\xi^m||^2 dK + [\Pi^m]$. We infer therefore from the last two inequalities and (3.94) that

$$\mathbb{E}\left[\int_\tau^T ||\sigma_s\xi_s^m||^2 dK_s + [\Pi^m]_T - [\Pi^m]_\tau \Big| \mathcal{F}_\tau\right] \leq 2e^{||\psi^m||_\infty}\Psi^{'}(||\psi^m||_\infty)\left(k_m^{'}\, ||\mu\cdot M||^2_{BMO} + k_m^{''} K_{BMO}\right).$$

Since τ is arbitrary, it follows from the above estimate that

$$\left|\left|\int_0^\cdot \xi^m dM\right|\right|_{BMO} + ||\Pi^m||_{BMO} \leq 2e^{||\psi^m||_\infty}\Psi^{'}(||\psi^m||_\infty)\left(k_m^{'}\, ||\mu\cdot M||^2_{BMO} + k_m^{''} K_{BMO}\right). \quad (3.95)$$

As $(\rho_m)_{m\in\mathbb{N}}$ converges to 0, the sequences $(k_m^{'})_{m\in\mathbb{N}}$ and $(k_m^{''})_{m\in\mathbb{N}}$ are bounded. Due to the first assertion, $||\psi^m||_\infty$ tends to 0 as $m \to +\infty$ and $\Psi^{'}(0) = 0$. Consequently, the right hand side of (3.95) goes to 0 as m tends to ∞ and (3.92) holds.
We can now prove b). For $m \in \mathbb{N}$, let $\gamma^m = \frac{1}{A_{U_m}(\widehat{X}^m)} - \frac{1}{c}$ and $\delta^m = \frac{1}{A_{U_m}(\widehat{X}^m)}$. Let $m \in \mathbb{N}$. As $\nu^m = \frac{1}{A_{U_m}(\widehat{X}^m)}\left(\mu + \xi^{L^m}\right)$ and $\nu^{L(1-c)} = \frac{1}{c}\left(\mu + \xi^{L(1-c)}\right)$,

$$\nu^m - \nu^{L(1-c)} = \gamma^m\left(\mu + \xi^{L(1-c)}\right) + \delta^m\left(\xi^m\right).$$

For $\tau \in \mathcal{T}(\mathbb{F})$, we have

$$\mathbb{E}\left[\int_\tau^T ||\sigma_s(\nu_s^m - \nu_s)||^2 dK_s \Big| \mathcal{F}_\tau\right] \leq 2||\gamma^m||^2_\infty \mathbb{E}\left[\int_\tau^T ||\sigma_s(\mu_s + \xi_s^{L(1-c)})||^2 dK_s \Big| \mathcal{F}_\tau\right]$$
$$+ \; 2||\delta^m||^2_\infty \mathbb{E}\left[\int_\tau^T ||\sigma_s\xi_s^m||^2 dK_s \Big| \mathcal{F}_\tau\right]$$
$$\leq 2||\gamma^m||_\infty \left|\left|\int_0^\cdot (\mu + \xi^{L(1-c)}) dM\right|\right|^2_{BMO} + 2||\delta^m||_\infty \left|\left|\int_0^\cdot \xi^m dM\right|\right|^2_{BMO}.$$

Now $||\gamma^m||_\infty$ converges to 0 as $A_{U_m} - \frac{1}{c}$ converges uniformly to 0. The sequence $(||\delta^m||_\infty)_{m\in\mathbb{N}}$ is bounded and thus by (3.92), $||\delta^m||_\infty\, ||\int_0^\cdot \xi^m dM||^2_{BMO}$ converges to 0 as $m \to +\infty$. From the above estimate, we deduce that b) holds.
For the proof of c), we rely on the continuity result given by Theorem 3.5.7. We recall that

$d\mathbb{Q}^\mu/d\mathbb{P} = Z^\mu_T$ with $Z^\mu = \mathcal{E}(-\mu\cdot M)$ and $dR = dM + d\langle M\rangle\mu$. Let $m\in\mathbb{N}$. Note that by Theorem 3.2.6, we have $\int_0^\cdot(\nu^m-\nu)dR \in BMO(\mathbb{Q}^\mu)$ and there exists $C_{BMO}>0$ depending only $||\mu\cdot M||_{BMO}$ such that

$$\left|\left|\int_0^\cdot(\nu^m-\nu)dR\right|\right|_{BMO(\mathbb{Q}^\mu)} \leq C_{BMO}\left|\left|\int_0^\cdot(\nu^m-\nu)dM\right|\right|_{BMO}.$$

We deduce from b) that as m tends to ∞,

$$\left|\left|\int_0^\cdot(\nu^m-\nu)dR\right|\right|_{BMO(\mathbb{Q}^\mu)} \to 0.$$

Now as $\widehat{X}^m/x_0 = \mathcal{E}\left(\int_0^\cdot \nu^m dR\right)$ and $\widehat{X}(1-c) = \mathcal{E}\left(\int_0^\cdot \nu dR\right)$, Theorem 3.5.7 and the convergence of $||\int_0^\cdot(\nu^m-\nu)dR||_{BMO(\mathbb{Q}^\mu)}$ yields that $||\widehat{X}^m/x_0-\widehat{X}(1-c)||_{\mathbb{H}_1(\mathbb{Q}^\mu)}$ converges to 0. Thus c) holds.

d). First we show that $||\widehat{Y}^m/\widehat{Y}^m_0-\widehat{Y}(1-c)/\widehat{Y}_0(1-c)||_{\mathbb{H}_1}$ tends to 0 as m goes to ∞. For $m\in\mathbb{N}$, $\widehat{Y}^m/\widehat{Y}^m_0 = \mathcal{E}\left(-\mu\cdot M+\Pi^{L^m}\right)$, $\widehat{Y}(1-c)/\widehat{Y}_0(1-c) = \mathcal{E}\left(-\mu\cdot M+\Pi^{L(1-c)}\right)$ and by (3.92),

$$||\Pi^{L^m}-\Pi^{L(1-c)}||_{BMO} = ||\Pi^m||_{BMO}\to 0 \text{ as } m\to+\infty.$$

Theorem 3.5.7 entails that $||\widehat{Y}^m/\widehat{Y}^m_0-\widehat{Y}(1-c)/\widehat{Y}_0(1-c)||_{\mathbb{H}_1}$ tends to 0 as $m\to\infty$.
It remains to show that as m goes to $+\infty$, $\mathcal{I}\left(\widehat{\mathbb{Q}}^m,\widehat{\mathbb{Q}}(1-c)\right)\to 0$. Let $m\in\mathbb{N}$. Clearly the measures $\widehat{\mathbb{Q}}^m$ and $\widehat{\mathbb{Q}}(1-c)$ are equivalent and by definition

$$\begin{aligned}\mathcal{I}\left(\widehat{\mathbb{Q}}^m,\widehat{\mathbb{Q}}(1-c)\right) &= \mathbb{E}^{\widehat{\mathbb{Q}}(1-c)}\left[\frac{d\widehat{\mathbb{Q}}^m}{d\widehat{\mathbb{Q}}(1-c)}\ln\frac{d\widehat{\mathbb{Q}}^m}{d\widehat{\mathbb{Q}}(1-c)}\right] = \mathbb{E}\left[\frac{d\widehat{\mathbb{Q}}^m}{d\mathbb{P}}\ln\frac{d\widehat{\mathbb{Q}}^m}{d\widehat{\mathbb{Q}}(1-c)}\right]\\ &= \mathbb{E}^{\widehat{\mathbb{Q}}^m}\left[\ln\widehat{Y}^m_T/\widehat{Y}^m_0-\ln\widehat{Y}_T(1-c)/\widehat{Y}_0(1-c)\right].\end{aligned}$$

Using the definitions of $\widehat{Y}^m$ and $\widehat{Y}(1-c)$, we have

$$\ln\widehat{Y}^m_T/\widehat{Y}^m_0-\ln\widehat{Y}_T(1-c)/\widehat{Y}_0(1-c) = \Pi^{L^m}_T-\frac{1}{2}\left\langle\Pi^{L^m}\right\rangle_T-\Pi^{L(1-c)}_T+\frac{1}{2}\left\langle\Pi^{L(1-c)}\right\rangle_T \tag{3.96}$$

The terms on the right hand side of (3.96) can be rewritten as follows:

$$\Pi^{L^m}_T-\frac{1}{2}\left\langle\Pi^{L^m}\right\rangle_T = \Pi^{L^m}_T-\left\langle\Pi^{L^m}\right\rangle_T+\frac{1}{2}\left\langle\Pi^{L^m}\right\rangle_T$$
$$-\Pi^{L(1-c)}_T+\frac{1}{2}\left\langle\Pi^{L(1-c)}\right\rangle_T = -\Pi^{L(1-c)}_T+\left\langle\Pi^{L(1-c)},\Pi^{L^m}\right\rangle_T-\left\langle\Pi^{L(1-c)},\Pi^{L^m}\right\rangle_T+\frac{1}{2}\left\langle\Pi^{L(1-c)}\right\rangle_T.$$

Since Π^{L^m} and $\Pi^{L(1-c)}$ are BMO martingales, Theorem 3.2.6 implies that

$$\Pi^{L^m}-\left\langle\Pi^{L^m}\right\rangle\in BMO(\widehat{\mathbb{Q}}^m) \text{ and } -\Pi^{L(1-c)}+\left\langle\Pi^{L(1-c)},\Pi^{L^m}\right\rangle\in BMO(\widehat{\mathbb{Q}}^m).$$

By integrating (3.96) w.r.t. $\widehat{\mathbb{Q}}^m$, one obtains the following reduced form of $\mathcal{I}\left(\widehat{\mathbb{Q}}^m,\widehat{\mathbb{Q}}(1-c)\right)$

$$\begin{aligned}\mathcal{I}\left(\widehat{\mathbb{Q}}^m,\widehat{\mathbb{Q}}(1-c)\right) &= \mathbb{E}^{\widehat{\mathbb{Q}}^m}\left[\frac{1}{2}\left\langle\Pi^{L^m}\right\rangle_T-\left\langle\Pi^{L(1-c)},\Pi^{L^m}\right\rangle_T+\frac{1}{2}\left\langle\Pi^{L(1-c)}\right\rangle_T\right]\\ &= \frac{1}{2}\mathbb{E}^{\widehat{\mathbb{Q}}^m}\left[\left\langle\Pi^{L^m}-\Pi^{L(1-c)}\right\rangle_T\right] = \mathbb{E}^{\widehat{\mathbb{Q}}^m}\left[\langle\Pi^m\rangle_T\right].\end{aligned}$$

Since $\sup_{m\geq N_\epsilon}||\Pi^{L^m}||^2_{BMO}\leq K_{BMO}$ by (3.90) and $\widehat{Y}^m_T/\widehat{Y}^m_0 = \mathcal{E}\left(-\mu\cdot M+\Pi^{L^m}\right)$ for every $m\in\mathbb{N}$, we infer from Theorem 3.2.6 that there exists $k>1$ and $\eta>0$ such that

$$\sup_{m\in N_\epsilon,\ \sigma\in\mathcal{T}(\mathbb{F})}\mathbb{E}\left[\left(\frac{\widehat{Y}^m_T}{\widehat{Y}^m_\tau}\right)^k\Bigg|\mathcal{F}_\tau\right] \leq \eta.$$

Let $r \in \mathbb{N}$ such that $\frac{r}{r-1} < k$ and $m \geq N_\epsilon$. Then using arguments as the one leading to (3.91) , we get

$$\mathbb{E}^{\widehat{\mathbb{Q}}^m}\left[\langle \Pi^m \rangle_T\right] \leq \left(\mathbb{E}\left[\left(\widehat{Y}_T^m/\widehat{Y}_0^m\right)^{\frac{r}{r-1}}\right]\right)^{\frac{r-1}{r}} \left(\mathbb{E}\left[\langle \Pi^m \rangle_T^r\right]\right)^{\frac{1}{r}} \leq \eta^{\frac{1}{k}}\left(r!||\Pi^m||_{BMO}^{2r}\right)^{\frac{1}{r}}.$$

Using (3.92), we see that $\mathbb{E}^{\widehat{\mathbb{Q}}^m}\left[\langle \Pi^m \rangle_T\right]$ converges to 0 as m tends to $+\infty$. We deduce therefore that $\mathcal{I}\left(\widehat{\mathbb{Q}}^m, \widehat{\mathbb{Q}}(1-c)\right)$ goes to 0 as $m \to +\infty$.

We consider the case $c \in (1, +\infty)$. The most important point for the case $c \in (0,1)$ was the boundedness of the sequence $\left(||\log L^m||_\infty + ||\log L(1-c)||_\infty\right)_{m\in\mathbb{N}}$. Once we show such a bound for $c \in (1, +\infty)$ under the BMO property of $\mu \cdot M$ the same arguments lead to the results. To achieve this, we observe that $\frac{c-1}{c} \geq 0$ since $c > 1$. Thus as Φ_{U_m} converges to $\frac{1}{2}\frac{c-1}{c}$, there exists $\gamma < \frac{1}{2c}$ and $N_\gamma > 0$ such that for all $m > N_\gamma$

$$0 \leq \Phi_{U_m} \leq \frac{1}{2}\frac{c-1}{c} + \gamma = \overline{\alpha_\gamma}.$$

Setting $b_\gamma = \frac{1}{1-2\overline{\alpha_\gamma}}$, U_m satisfies $(G_{1,b_\gamma,1})$ for every $m \geq N_\gamma$ by Remark 3.4.4. We infer therefore from Proposition 3.4.21 that there exists $K_\infty > 0$ depending only on $||\mu \cdot M||_\infty$ such that

$$\sup_{m \geq N_\gamma}\left(||\log L^m||_\infty + ||\log L(1-c)||_\infty\right) \leq K_\infty.$$

This completes the proof. □

3.5.2 The limit $c = 1$

In this case, $\Phi_{U_m} \to 0$ and $A_{U_m} \to 1$ uniformly as $m \to +\infty$. For the logarithmic utility $U = \log$, we have $A_U = 1$ and $\Phi_U = 0$. Following Remark 3.5.5, we consider the logarithmic utility maximization problem

$$u_{\log}(x_0) := \sup_{\pi \in \mathcal{A}(x_0)} \mathbb{E}\left[\log X_T^\pi\right]. \tag{3.97}$$

Recall that $u_{\log}(x_0) < +\infty$ if Assumption 3.4.7 is satisfied or if $\mu \cdot M$ is a BMO martingale[8]. Note that for logarithmic utility, the opportunity process L equals 1. Therefore from the canonical decomposition (3.12), $Z^L = 0$ and $N^L = 0$. With $Z^\mu = \mathcal{E}(-\mu \cdot M)$, we infer from Theorem 3.3.5 that the corresponding optimal trading strategy ν, optimal wealth process $\widehat{X}$ and dual optimizer Y have the precise formulas

$$\nu = \mu, \quad \widehat{X} = x_0/Z^\mu \text{ and } \widehat{Y} = u'(x_0)Z^\mu. \tag{3.98}$$

Theorems 3.5.13 and 3.5.15 give the convergence of the sequences $(\nu^m)_{m\in\mathbb{N}}, (\widehat{X}^m)_{m\in\mathbb{N}}$ and $(\widehat{Y}^m)_{m\in\mathbb{N}}$.

Theorem 3.5.13. *Suppose that Assumption 3.4.7 holds. Let $(U_m)_{m\in\mathbb{N}}$ be a sequence of utility functions satisfying Assumption 3.5.1, $Z^\mu = \mathcal{E}(-\mu \cdot M)$, and $(\nu^m)_{m\in\mathbb{N}}, (\widehat{X}^m)_{m\in\mathbb{N}}, (\widehat{Y}^m)_{m\in\mathbb{N}}$ defined by (3.71). Then as $m \to +\infty$:*

C1. $||\nu^m - \nu||_{\mathcal{H}^2} \to 0,$

[8] Indeed, by Proposition 2.2.7, there exists $p \in (0,1)$ such that Z^μ satisfies $(A_{\frac{1}{p}})$. In particular, $\mathbb{E}\left[\left(Z_T^\mu\right)^{\frac{p}{p-1}}\right] < +\infty$ and $\overline{u}_p < +\infty$ by Remark 2.3.1. Now there exists $\overline{z} > 0$ such that $\log z \leq z^p$ for all $z > \overline{z}$. The finiteness of $u(x_0)$ follows from that of $\overline{u}_p$.

C2. $\widehat{X}^m \to x_0/Z^\mu$ *in* $\mathcal{S}_0$ *and* $\widehat{Y}^m/\widehat{Y}_0 \to Z^\mu$ *in* $\mathcal{S}_0$.

Proof. Let $f_{\log}(t,z) = \frac{1}{2}||\sigma_t z||^2$, $(t,z) \in [0,T] \times \mathbb{R}^n$. Clearly, $(0,0,0)$ is the unique solution to the BSDE$(f_{\log}, 0)$ in the space $\varXi \times \mathcal{H}^2 \times \mathcal{M}^2$. Using F_{U_m} given by (3.70) for $m \in \mathbb{N}$, we have

$$\mathbb{E}\left[\int_0^T |F_{U_m}(t, \widehat{X}_t^m, 0) - f_{\log}(t,0)|dK_t\right] = \mathbb{E}\left[\int_0^T |\Phi_{U_m}(\widehat{X}_t^m)| \times ||\sigma_t\mu_t||^2 dK_t\right] \le ||\Phi_{U_m}(\widehat{X}^m)||_\infty ||\mu||^2_{\mathcal{H}^2}.$$

The above inequality combined with the uniform convergence of Φ_{U_m} to 0 as $m \to +\infty$ give

$$\lim_{m\to+\infty} \int_0^T |F_{U_m}(t, \widehat{X}_t^m, 0) - f_{\log}(t,0)|dK_t = 0 \text{ in probability.}$$

Using similar arguments as in Lemma 3.5.9, with $\xi^{L(1-c)}$ replaced by 0 and $\Pi^{L(1-c)}$ by 0, one sees that $\lim_{m\to+\infty} ||\xi^{L^m}||_{\mathcal{H}^2} + ||\Pi^{L^m}||_{\mathcal{M}^2} = 0$. Similar arguments as in the proof of Theorem 3.5.8 lead to C1. and C2. □

Remark 3.5.14. *Theorem 3.5.13 is a generalization of [MW13, Theorem 3.8] for* $U_m(z) = \frac{z^{p_m}}{p_m}, z > 0$ *and* $(p_m)_{\in\mathbb{N}} \subseteq (-\infty, 0) \cup (0,1)$ *a sequence converging to* 0*. In this particular case of power utilities, the continuity assumption on the filtration and Assumption 3.4.7 can be omitted [Nut12c, Theorem 3.4]. This is mainly possible due to the monotonicity of* $L(p)$ *w.r.t.* p *which yields pointwise convergence of* $(L(p_m))_{m\in\mathbb{N}}$ *and also allows for a uniform localization of the sequence* $(L(p_m))_{m\in\mathbb{N}}$ *in the space* $\mathcal{S}^\infty$*. We do not have such a monotonicity property in the general setup and our reliance on Theorem 3.5.6 for the convergence of BSDEs results is at the basis of the restrictions in Theorem 3.5.13.*

Theorem 3.5.15. *Suppose that* $\mu \cdot M \in BMO$*. We keep the notation of Theorem 3.5.13. In addition for* $m \in \mathbb{N}$*, let* $d\widehat{\mathbb{Q}}^m/d\mathbb{P} = \widehat{Y}_T^m/\widehat{Y}_0^m$ *and* $d\mathbb{Q}^\mu/d\mathbb{P} = Z_T^\mu$*. Then as* $m \to +\infty$*:*

a) $||\log L^m||_\infty \to 0$,

b) $||(\nu^m - \mu)\cdot M||_{BMO} \to 0$,

c) $||\widehat{X}^m \to x_0/Z^\mu||_{\mathbb{H}_1(\mathbb{Q}^\mu)} \to 0$,

d) $||\widehat{Y}^m/\widehat{Y}_0^m - Z^\mu||_{\mathbb{H}_1} + \mathcal{I}(\widehat{\mathbb{Q}}^m, \mathbb{Q}^\mu) \to 0$.

Proof. It suffices to show that $(||\log L^m||_\infty)_{m\in\mathbb{N}}$ is uniformly bounded from a certain rank and apply the same arguments as in Theorem 3.5.11 replacing f_{1-c} by $f_{\log}$, $\xi^{L(1-c)}$ by 0 and $\Pi^{L(1-c)}$ by 0.
As $\mu \cdot M \in BMO$, $\delta = b(\mu \cdot M) > 0$. We choose $\epsilon \in (0, \frac{1}{2})$ such that $\delta > \kappa(2\epsilon)$. Since $\Phi_{U_m} \to 0$ as $m \to +\infty$, there exists $N_\epsilon \in \mathbb{N}$, such that for every $m \ge N_\epsilon, -\epsilon \le \Phi_{U_m} \le \epsilon$. Let $a_\epsilon = \frac{1}{1+2\epsilon} \in (0,1)$. Then $b(\mu\cdot M) = \delta > \kappa(2\epsilon) = \kappa(\frac{1-a_\epsilon}{a_\epsilon})$. We deduce from Proposition 3.4.21 that there exists $K_\infty > 0$ such that $\sup_{m\ge N_\epsilon} ||\log L^m||_\infty \le K_\infty$. □

Remark 3.5.16. *Assertion d) has been obtained in [MT03b] for* $U^m(z) = -mz^{-\frac{1}{m}}, z > 0, m \in \mathbb{N}$*. In this case, note that for each* $m \in \mathbb{N}$*,* $L^m = L(-\frac{1}{m})$ *,*$\Phi_{U_m} = \frac{1}{2}\frac{1}{1+m} \downarrow 0$ *and by Proposition 3.4.21 the BMO property of* $\mu \cdot M$ *is sufficient to have* $\sup_{m\in\mathbb{N}} ||\log L^m||_\infty < +\infty$*. However for an arbitrary sequence* $(U_m)_{m\in\mathbb{N}}$*,* $(\Phi_{U_m})_{m\in\mathbb{N}}$ *might not be monotone nor has a steady positive sign. Due to these possibilities, the sufficiency of the BMO property of* $\mu \cdot M$ *is not straightforward and the choice of* ϵ *in the above proof is necessary to guarantee the existence of a uniform bound for* $(||\log L^m||_\infty)_{m\in\mathbb{N}}$.

3.5.3 The limit $c = +\infty$

Here we have $\Phi_{U_m} \to \frac{1}{2}$ and $A_{U_m} \to +\infty$ as $m \to +\infty$. As mentioned earlier, the limits of the optimizers are related to the exponential utility maximization problem which we will now recall. First, let us introduce the following set

$$\mathcal{M}^{ent}(S) = \{\mathbb{Q} \in \mathcal{M}^e(S) \mid \mathcal{I}(\mathbb{Q}, \mathbb{P}) < +\infty\}. \tag{3.99}$$

Throughout this section, we identify $\mathbb{Q} \in \mathcal{M}^{ent}(S)$ with its density process $Z^{\mathbb{Q}}$ and we assume that

$$\mathcal{M}^{ent}(S) \neq \emptyset.$$

The exponential utility function is given by $U(z) = -\exp(-z), z \in \mathbb{R}$. For the given initial capital x_0, the investor aims to maximize his expected utility from terminal wealth, i.e.

$$\sup_{\pi \in \Theta} \mathbb{E}\left[-\exp\left(-x_0 - (\pi \cdot R)_T\right)\right], \tag{3.100}$$

where $\Theta = \{\pi \in \mathcal{L}(R) \text{ s.t. } \pi \cdot R \text{ is a } \mathbb{Q}-\text{martingale } \forall \mathbb{Q} \in \mathcal{M}^{ent}(S)\}$. A trading strategy attaining the sup in (3.100) will be refereed to as optimal. Such a strategy exists (see [KS02, Theorem 2.1]) and will be denoted by $\nu^{\exp}$. Related to (3.100) is the following minimization problem

$$\min_{\mathbb{Q} \in \mathcal{M}^{ent}(S)} \mathcal{I}(\mathbb{Q}, \mathbb{P}). \tag{3.101}$$

By the results in [Fri00, Theorems 2.1 and 2.2], the minimization problem (3.101) admits a unique solution $\mathbb{Q}^{\exp}$ known as the the *minimal entropy martingale measure*, see [Fri00, GR02]. We consider the processes $L^{\exp}$ and $\widetilde{V}$ defined as follows

$$\widetilde{V}_t = \operatorname*{ess\,inf}_{\mathbb{Q} \in \mathcal{M}^{ent}(S)} \mathbb{E}\left[\frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}} \log \frac{Z_T^{\mathbb{Q}}}{Z_t^{\mathbb{Q}}} \Big| \mathcal{F}_t\right], \quad t \in [0, T] \tag{3.102}$$

$$L_t^{\exp} = \operatorname*{ess\,inf}_{\pi \in \Theta} \mathbb{E}\left[\exp\left(-\int_t^T \pi_u dR_u\right) | \mathcal{F}_t\right], \quad t \in [0, T]. \tag{3.103}$$

Clearly $\widetilde{V}$ is the dynamic value process associated to (3.101). The process $L^{\exp}$ is a reduced form of the value process associated to (3.100). The following proposition collects some properties of $L^{\exp}$.

Proposition 3.5.17. *i) $L^{\exp}$ is a submartingale with terminal value 1 and $L^{\exp} = e^{-\widetilde{V}}$.*

ii) There exists $Z^{L^{\exp}} \in \mathcal{L}(M)$ and $N^{L^{\exp}}$ a local martingale orthogonal to M such that for $t \in [0, T]$

$$dL^{\exp} = Z_t^{L^{\exp}} dM_t + dN_t^{L^{\exp}} + \frac{1}{2} L_t^{\exp} \left(\mu_t + \frac{Z_t^{L^{\exp}}}{L_t^{\exp}}\right)^\top d\langle M \rangle_t \left(\mu_t + \frac{Z_t^{L^{\exp}}}{L_t^{\exp}}\right). \tag{3.104}$$

iii) The optimal trading strategy $\nu^{\exp}$ and the density process of $\mathbb{Q}^{\exp}$ are given by

$$\nu^{\exp} = \mu + \xi^{L^{\exp}} \text{ and } Y^{\exp} = \mathcal{E}\left(-\mu \cdot M + \Pi^{L^{\exp}}\right), \tag{3.105}$$

where

$$\xi^{L^{\exp}} = \frac{Z^{L^{\exp}}}{L^{\exp}} \text{ and } N^{L^{\exp}} = \int_0^{\cdot} \frac{1}{L^{\exp}} dN^{L^{\exp}}. \tag{3.106}$$

iv) If Assumption 3.4.7 holds, then $\log L^{\exp} \in \Xi$.

v) $\log L^{\exp}$ *is bounded if* $\mu \cdot M \in BMO$.

Proof. i) Clearly $L_T^{\exp} = 1$ by definition. The submartingale property of $L^{\exp}$ is a consequence of dynamic programming principle, see [Nut12c, Lemma 6.5] or [MT03b, Proposition 2.1]. Regarding the equality $L^{\exp} = e^{-\widetilde{V}}$, we refer to [MS05, Proposition 2] or [MT03b, Proposition 2.3]. The assertions ii) and iii) have been obtained in [MT03b, Theorem 3.1].
iv) Now assume that $\mathbb{F}$ is continuous and Assumption 3.4.7 holds. To show that $\log L^{\exp} \in \Xi$, we use the equality $L = e^{-\widetilde{V}}$ and (3.102). We recall that $Z^\mu = \mathcal{E}(-\mu \cdot M), dR = dM + d\langle M\rangle\mu$ and $d\mathbb{Q}^\mu/d\mathbb{P} = Z_T^\mu$. Observe that $\mathbb{Q}^\mu \in \mathcal{M}^e(S)$. First we show that $Z^\mu \in \mathcal{M}^{ent}(S)$. To this end, note that for $t \in [0,T]$

$$\log(Z_T^\mu/Z_t^\mu) = -\int_t^T \mu_s dM_s - \frac{1}{2}\int_t^T \mu_s^\top d\langle M\rangle_s \mu_s = -\int_t^T \mu_s dR_s + \frac{1}{2}\int_t^T \mu_s^\top d\langle M\rangle_s \mu_s.$$

By Girsanov's theorem, $\mu \cdot R$ is a $\mathbb{Q}^\mu$-local martingale. An application of Hölder's inequality and Proposition 3.4.9 give

$$\mathbb{E}^{\mathbb{Q}^\mu}\left[\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\right] = \left(\mathbb{E}\left[(Z_T^\mu)^2\right]\right)^{\frac{1}{2}} \left(\mathbb{E}\left[\left(\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\right)^2\right]\right)^{\frac{1}{2}} < +\infty. \tag{3.107}$$

We infer from (3.107) and Burkholder-Davis-Gundy's inequalities that $\mu \cdot R$ is a $\mathbb{Q}^\mu$ martingale. Thus for $t \in [0,T]$

$$\mathbb{E}\left[\frac{Z_T^\mu}{Z_t^\mu}\log\frac{Z_T^\mu}{Z_t^\mu}\bigg|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^\mu}\left[-\int_t^T \mu_s dR_s + \frac{1}{2}\int_t^T \mu_s^\top d\langle M\rangle_s \mu_s\bigg|\mathcal{F}_t\right] = \mathbb{E}^{\mathbb{Q}^\mu}\left[\int_t^T \mu_s^\top d\langle M\rangle_s \mu_s\bigg|\mathcal{F}_t\right]. \tag{3.108}$$

Taking $t = 0$ in the above inequality, we see that $Z^\mu \in \mathcal{M}^{ent}(S)$. As $|\log L| = \widetilde{V}$ and $Z^\mu \in \mathcal{M}^{ent}(S)$, (3.102) implies that $|\log L_t^{\exp}| \le \mathbb{E}\left[\frac{Z_T^\mu}{Z_t^\mu}\log\frac{Z_T^\mu}{Z_t^\mu}|\mathcal{F}_t\right], t \in [0,T]$. Hence, (3.108) yields

$$\sup_{t\in[0,T]} |\log L_t^{\exp}| \le \frac{1}{2}\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{Q}^\mu}\left[\int_t^T \mu_s^\top d\langle M\rangle_s \mu_s\bigg|\mathcal{F}_t\right] \le \frac{1}{2}\sup_{t\in[0,T]} \mathbb{E}^{\mathbb{Q}^\mu}\left[\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\bigg|\mathcal{F}_t\right].$$

The filtration $\mathbb{F}$ being continuous, all local martingale are continuous. Consequently for $r > 0$, exchanging sup and exp, and applying Jensen's inequality, we get

$$\begin{aligned}\exp\left(r\sup_{t\in[0,T]}|\log L_t^{\exp}|\right) &\le \left(\exp\left(\sup_{t\in[0,T]}\mathbb{E}\left[\frac{1}{2}\int_0^T \mu_s^\top d\langle M\rangle_s \mu_s\bigg|\mathcal{F}_t\right]\right)\right)^r \\ &\le \sup_{t\in[0,T]}\left(\mathbb{E}^{\mathbb{Q}^\mu}\left[\exp\left(\frac{1}{2}\int_0^T \mu_s^\top d\langle M\rangle \mu_s\right)\bigg|\mathcal{F}_t\right]\right)^r.\end{aligned}$$

We define $\zeta_T = \exp\left(\frac{1}{2}\int_0^T \mu_s^\top d\langle M\rangle \mu_s\right)$ and $\zeta_t = \mathbb{E}^{\mathbb{Q}^\mu}\left[\zeta_T|\mathcal{F}_t\right],\ t \in [0,T]$. Using Doob's maximal inequalities and Proposition 3.4.9, one verifies that $\zeta \in \mathcal{S}^k(\mathbb{Q}^\mu) \cap \mathcal{S}^k$ for all $k > 1$. Taking expectations in the previous inequality, the integrability of ζ ensures that

$$\mathbb{E}\left[\exp\left(r\sup_{t\in[0,T]}|\log L_t^{\exp}|\right)\right] < +\infty.$$

Since r is arbitrary, we deduce that $\log L^{\exp} \in \Xi$.

Suppose that $\mu \cdot M \in BMO$. We keep the notation in iv). As $\mu \cdot M \in BMO$, Theorem 3.2.6 entails that $\mu \cdot R$ is a $\mathbb{Q}^\mu$-martingale. Similar arguments as in iv) show that $\mathbb{Q}^\mu \in \mathcal{M}^{ent}(S)$ and (3.108) holds. Using once more the equality $L^{\exp} = e^{-\widetilde{V}}$, (3.102) and (3.108), we have for $t \in [0,T]$

$$\log L^{\exp}_t = -\widetilde{V}_t \geq -\mathbb{E}\left[\frac{Z^\mu_T}{Z^\mu_t}\log\frac{Z^\mu_T}{Z^\mu_t}\Big|\mathcal{F}_t\right] \geq -\frac{1}{2}\mathbb{E}^{\mathbb{Q}^\mu}\left[\int_t^T \mu_s^\top d\langle M\rangle_s \mu_s\Big|\mathcal{F}_t\right] \geq -\frac{1}{2}||\mu\cdot R||^2_{BMO(\mathbb{Q}^\mu)}.$$

We deduce that $\log L^{\exp}$ is bounded from below. Since $L^{\exp}_T = 1$ and $L^{\exp}$ is a submartingale, we have $\log L^{\exp} \leq 0$. □

With $(\xi^{L^{\exp}}, \Pi^{L^{\exp}})$ defined by (3.106), an application of Itô's formula to $\log L^{\exp}$ using (3.104) gives

$$d\log L^{\exp}_t = \xi^{L^{\exp}}_t dM_t + d\Pi^{L^{\exp}}_t - f_{\exp}(t, \xi^{L^{\exp}}_t)dK_t,\ t\in[0,T],\ \log L^{\exp}_T = 0, \tag{3.109}$$

where $f_{\exp}$ is the driver defined as follows:

$$f_{\exp}(t,z) \quad = \quad -\frac{1}{2}||\sigma_t\mu_t||^2 - (\sigma_t\mu_t)^\top(\sigma_t z),\ (t,z)\in[0,T]\times\mathbb{R}^n.$$

The following corollary of Proposition 3.5.17 identifies $\left(\log L^{\exp}, \xi^{L^{\exp}}, \Pi^{L^{\exp}}\right)$ as the unique solution to the BSDE $(f_{\exp}, 0)$ under certain conditions on the market price of risk μ.

Corollary 3.5.18. *Let $(\xi^{L^{\exp}}, \Pi^{L^{\exp}})$ be defined by (3.106). If Assumption 3.4.7 holds, then the triplet $\left(\log L^{\exp}, \xi^{L^{\exp}}, \Pi^{L^{\exp}}\right)$ is the unique solution to the BSDE $(f_{\exp}, 0)$ in the space $\varXi \times \mathcal{H}^2 \times \mathcal{M}^2$. If $\mu \cdot M$ is a BMO martingale, then $\left(\log L^{\exp}, \xi^{L^{\exp}}, \Pi^{L^{\exp}}\right)$ is the unique solution to the BSDE $(f_{\exp}, 0)$ with bounded first component.*

Proof. The driver $f_{\exp}$ is convex and Lipschitz in the control variable z. Under Assumption 3.4.7, $\log L^{\exp} \in \varXi$ by Proposition 3.5.17. We infer from Theorem 3.4.8 that $\left(\log L^{\exp}, \xi^{L^{\exp}}, \Pi^{L^{\exp}}\right)$ belongs to $\varXi \times \mathcal{H}^2 \times \mathcal{M}^2$ and it is the unique solution to the BSDE $(f_{\exp}, 0)$ in the latter space. Now if $\mu \cdot M \in BMO$, $\log L^{\exp}$ is bounded by Proposition 3.5.17. The uniqueness of the solution follows from classical linearization and change of measures techniques, see [HIM05, Tev08, Mor09a]. □

To obtain the limits of our sequence of optimizers, we observe that as $(\Phi_{U_m})_{m\in\mathbb{N}}$ converging uniformly to $\frac{1}{2}$, the sequence of drivers $F_{U_m} \to f_{\exp}$ as $m \to +\infty$ pointwise. Having established strong integrability properties of the solutions, we expect their martingale parts to converge in their respective space and thus the limits of optimizers to be related to the processes $\nu^{\exp}, Y^{\exp}$ and $\log L^{\exp}$. The following two theorems give a precise formulation of these facts.

Theorem 3.5.19. *Suppose that Assumption 3.4.7 holds. Let $(U_m)_{m\in\mathbb{N}}$ be a sequence of utility functions satisfying Assumption 3.5.1 with $c = +\infty$. Let $(\nu^m)_{m\in\mathbb{N}}, (\widehat{X}^m)_{m\in\mathbb{N}}$ and $\widehat{Y}^m)_{m\in\mathbb{N}}$ be defined by (3.71). Let $\nu^{\exp}$ and $Y^{\exp}$ be defined by (3.105). Then as $m \to +\infty$:*

F1. $||\nu^m||_{\mathcal{H}^2} + ||A_{U_m}(\widehat{X}^m)\nu^m - \nu^{\exp}||_{\mathcal{H}^2} \to 0,$

F2. $\widehat{X}^m/x_0 \to 1$ *in* $\mathcal{S}_0$ *and* $\widehat{Y}^m/\widehat{Y}^m_0 \to \widehat{Y}^{\exp}$ *in* $\mathcal{S}_0$.

Proof. Due to Corollary 3.5.18, similar arguments as in the proof of Lemma 3.5.9 yield

$$\lim_{m\to+\infty} ||\xi^{L^m} - \xi^{L^{\exp}}||_{\mathcal{H}^2} + ||\Pi^{L^m} - \Pi^{L^{\exp}}||_{\mathcal{M}^2} = 0.$$

Let $m \in \mathbb{N}$. Then $\nu^m = \frac{1}{A_{U_m}(\widehat{X}^m)}(\mu + \xi^{L^m})$. As $\sup_{m\in\mathbb{N}} ||\xi^{L^m}||_{\mathcal{H}^2} < +\infty$ and $1/A_{U_m}$ tends to 0 uniformly, we deduce that $||\nu_m||_{\mathcal{H}^2}$ tends to 0 as $m \to +\infty$. With $\nu^{\exp}$ given by (3.105), we have

$$A_{U_m}(\widehat{X}^m)\nu^m - \nu^{\exp} = \xi^{L^m} - \xi^{L^{\exp}}.$$

The assertion F1. is thus a consequence of the fact that $\lim_{m\to+\infty} ||\xi^{L^m} - \xi^{L^{\exp}}||_{\mathcal{H}^2} = 0$. To show F2. one proceeds as in the proof of Theorem 3.5.19 replacing $\xi^{L(1-c)}$ by $\xi^{L^{\exp}}$ and $\Pi^{L(1-c)}$ by $\Pi^{L^{\exp}}$. □

Remark 3.5.20.
- *The fact that the optimal trading strategy converges to 0 as the relative risk aversion goes to 0 has been obtained in [GT11, Theorem 2.6] in the setting of a complete market model and with bounded market price risk μ using PDE methods. For $(U_m)_{m\in\mathbb{N}}$ of power type, a convergence in incomplete markets and without Assumption 3.4.7 has been obtained in [Nut12c, Theorem 3.1]. Theorem 3.5.19 is thus the most general result for general utility functions.*
- *Regarding the convergence of the rescaled optimal strategy, it has been observed in [Nut12c, Theorem 3.2] for $(U_m)_{m\in\mathbb{N}}$ of power type without Assumption 3.4.7. To the best of your knowledge, the extension to utility functions of arbitrary type is new in the literature.*

Theorem 3.5.21. *We keep the notation of Theorem 3.5.19. Assume that $\mu \cdot M \in BMO$. Then as $m \to +\infty$, the following hold:*

a) $|| \log L^m - \log L^{\exp}||_\infty \to 0$,

b) $||\nu^m \cdot M||_{BMO} + ||(A_{U_m}(\widehat{X}^m)\nu^m - \nu^{\exp}) \cdot M||_{BMO} \to 0$,

c) $||\widehat{X}^m/x_0 - 1||_{\mathbb{H}_1(\mathbb{Q}^\mu)} \to 0$,

d) $||\widehat{Y}^m/\widehat{Y}_0 - Y^{exp}||_{\mathbb{H}_1} + \mathcal{I}(\widehat{\mathbb{Q}}^m, \mathbb{Q}^{exp}) \to 0$.

Proof. Again the proof follows the same line of argument as the one of Theorem 3.5.11 once we establish that the BMO property of $\mu \cdot M$ leads to a uniform bound of the sequence $(\log L^m)_{m\in\mathbb{N}}$. Observe that as Φ_{U_m} converges to $\frac{1}{2}$ and $\Phi_{U_m} < \frac{1}{2}$, we can assume that for each $m \in \mathbb{N}$

$$0 \le \Phi_{U_m} \le \frac{1}{2} - \epsilon_m, \tag{3.110}$$

where $(\epsilon_m)_{m\in\mathbb{N}}$ is a sequence of positive integers in $(0, \frac{1}{2})$ converging to 0. By Remark 3.4.4, the above inequality entails that U_m satisfies $(G_{1,b_m,1})$ with $b_m = \frac{1}{1-2\epsilon_m}$ for each $m \in \mathbb{N}$. We deduce from Proposition 3.4.21 that the sequence $(||\log L^m||_\infty)_{m\in\mathbb{N}}$ is uniformly bounded. As $\log L^{\exp}$ is bounded by Proposition 3.5.17, we have

$$\sup_{m\in\mathbb{N}} (||\log L^m||_\infty + ||\log L^{\exp}||_\infty) < +\infty.$$

To obtain the assertions, we replace $f_{1-c}, \Pi^{L(1-c)}, \xi^{L(1-c)}$ respectively by $f_{\exp}, \Pi^{L^{\exp}}, \xi^{L^{\exp}}$ in the proof of Theorem 3.5.11. □

Remark 3.5.22. *Assertion d) in Theorem 3.5.21 has been previously obtained in [MT03b, Corollaries 4.1 and 4.2] for $U_m(z) = -\frac{z^{-m}}{m}, z > 0$ and $m \in \mathbb{N}$. A nonstandard example is given by $U_m(z) = -e^{-mz}z^{-m}, z > 0, m \in \mathbb{N}$*

4. Optimal stopping problem in a progressively enlarged filtration : a two step decomposition approach

4.1 Introduction

A common risk embedded in most financial contracts is the *default risk*, i.e. the risk that a party involved in a financial contract does not meet its obligations. Default risk affects parameters in the financial markets such as the prices of traded securities, rating of institutions, the infusion or redrawal of capital by investors and above all, panic sales or purchases of financial agents exposed to the risk. Due to the increased interconnectedness of financial institutions and the contagion effect of default risk, the magnitude of the impact of default can be large so as to affect the whole economy as seen in the 2008's mortgage crisis. This impact has led to an increase interest in the literature in the study of control problems faced by financial agents in the presence of default risk. In particular, one is interested in providing a description of their actions before and after the default event. Such problems include for examples the pricing of defaultable claims [EKJJ15a], mean-variance hedging of defaultable claims [CGN15] and portfolio optimization with defaultable securities [JP11, Pha10, JKP13, IJL16]. Our aim in this chapter is to provide a similar analysis for optimal stopping problem [EK81, KQ12] with a reward ζ that is subject to a default event modeled by a random time τ.

Our underlying filtration $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ modeling the global market information flow including the default event is given by the *progressive enlargement* of a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ by the random time τ, i.e $\mathcal{G}_t = \cap_{s>t}\left(\mathcal{F}_s \vee \sigma\left(1_{\{\tau\leq u\}}, u \leq s\right)\right),\ t \geq 0$. We assume the density hypothesis on the conditional distribution of τ given $\mathbb{F}$. The density hypothesis introduced in [Jac85] for the study of the *initial enlargement* of $\mathbb{F}$ with τ denoted by $\mathbb{G}^\tau = (\mathcal{G}^\tau_t)_{t\geq 0}$ where $\mathcal{G}^\tau_t = \cap_{s>t}\left(\mathcal{F}_s \vee \sigma(\tau)\right),\ t \geq 0$ has now been adopted for the study of the filtration $\mathbb{G}$, see [JLC09b, EKJJ10, CJZ13, JLC09a]. A particular feature of the density hypothesis shown in [EKJJ10] is that it leads to formulas for the conditional expectation w.r.t. $\mathcal{G}_t$ on the after default event $\{t \geq \tau\}$ and thus is suitable for the analysis of after default events. Another feature of the density hypothesis is the *optional splitting formula* obtained in [Son14] which asserts that every $\mathbb{G}$-optional process ψ is of the form

$$\psi_t \quad = \quad \psi^{\mathrm{b}}_t 1_{\{t<\tau\}} + \psi^{\mathrm{d}}_t(\tau)1_{\{t\geq\tau\}},\ t \geq 0,$$

where ψ^{b} is $\mathcal{O}(\mathbb{F})$-measurable, ψ^{d} is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R})$-measurable and $\mathcal{O}(\mathbb{F})$ denotes the σ-field of $\mathbb{F}$-optional sets. Exploiting the above features in the context where $\mathbb{F}$ is a Brownian filtration, in [JP11] the authors showed that optimal investment problems in the filtration $\mathbb{G}$ can be reduced to two sub optimal investment problems in the filtration $\mathbb{F}$: an after default problem parametrized by the occurrence of default and a global before default problem containing the latter. The optimal trading strategy is then obtained as the pasting at time τ of the optimal strategies resulting from the sub-control problems in the filtration $\mathbb{F}$. This yields a better understanding of the optimal strategy compared to the more direct approach in the filtration $\mathbb{G}$ using the dynamic programming principle or convex duality methods [KS99]. This approach reducing the optimization investment problem in the filtration $\mathbb{G}$ into two weakly coupled investment problems in the filtration $\mathbb{F}$ is now known in the literature as the *decomposition approach* due to the two step procedures it involves. It extends naturally to the setting of multiple default events [Pha10, JKP13] and has been applied successfully to address other stochastic control problems such as the mean-variance hedging [CGN15] and controller-stopper problems [BZ14]. A corresponding decomposition approach has been developed in [KL12, ABSEL10] for the solvability of backward stochastic differential equations (BSDEs) in the filtration $\mathbb{G}$.

We will adopt a similar approach as in [JP11] for the optimal stopping problem with reward ζ which consists in splitting the stopping problem into an after-default problem and a global before default problem. However, rather than focusing on the static value of the optimal stopping problem, we will look at the dynamic value process which corresponds to the Snell envelope V of the reward ζ. The Snell envelope being the main tool to address optimality and satisfying the optional splitting formula $V_t = V_t^{\mathrm{b}} 1_{\{t<\tau\}} + V_t^{\mathrm{d}}(\tau) 1_{\{t\geq\tau\}}$, $t \geq 0$, our approach will consist in providing the explicit expressions of V^{b} and $V^{\mathrm{d}}(\tau)$ which we link to optimal stopping problems in the filtrations $\mathbb{F}$ and $\mathbb{G}^\tau$ corresponding respectively to the optimal stopping problems to be solved before and after τ, the former depending on the latter. Building on the knowledge of the optimal stopping problems to be addressed before and after τ, we obtain as well a complete characterization of optimal stopping times before and after τ. The fact that the optimal stopping problem to be addressed after τ is in the filtration $\mathbb{G}^\tau$ is consistent with the fact that the filtrations $\mathbb{G}$ and $\mathbb{G}^\tau$ *coincide after* τ (see Definition 4.2.9) as shown in [KLP13]. We illustrate the importance of our approach by deriving an explicit formula for hedging strategies against American defaultable contingent claims in a market consisting of a default free asset and a defaultable zero coupon bond. In Chapter 5, we rely on the link between optimal stopping problems and reflected backward stochastic differential equations (RBSDEs) to provide a similar decomposition approach for the solvability of RBSDEs and new existence results.

The rest of this chapter is structured as follows. In Section 4.2, we set up the notation. We also review and provide some results on filtration enlargements. We present our decomposition approach for the optimal stopping problem in Section 4.3. In Section 4.4, we present an application of the decomposition approach by providing optional splitting formulas for hedging strategies for defaultable American contingent claims.

4.2 Filtration enlargements

We set $\mathbb{R}^+ = [0, +\infty[$. We fix some generic notation. Let $(\Sigma, \mathbb{A}, \mathbb{Q})$ be a complete probability space. We denote by $\mathbb{E}^{\mathbb{Q}}$ the expectation w.r.t. to $\mathbb{Q}$. Let $\mathbb{H} = (\mathcal{H}_t)_{t\geq 0}$ be a filtration of sub-σ-algebras of $\mathbb{A}$ and $\mathcal{H}_\infty = \bigvee_{t\in\mathbb{R}^+} \mathcal{H}_t$. We will denote by $\mathcal{P}(\mathbb{H})$ (resp. $\mathcal{O}(\mathbb{H})$) the σ-field of $\mathbb{H}$-predictable sets (resp. $\mathbb{H}$-optional sets) on $\mathbb{R}^+ \times \Omega$ and $\mathcal{T}(\mathbb{H})$ the set of all $\mathbb{H}$-stopping times. For $\nu, \gamma \in \mathcal{T}(\mathbb{H})$ with $\nu \leq \gamma$ $\mathbb{Q}$-a.s., $\mathcal{T}_{\nu,\gamma}(\mathbb{H})$ denotes the set of $\mathbb{H}$-stopping times σ such that $\mathbb{Q}(\nu \leq \sigma \leq \gamma) = 1$. For simplicity, we note $\mathcal{T}_{0,\nu}(\mathbb{H}) = \mathcal{T}_\nu(\mathbb{H})$ for $\nu \in \mathcal{T}(\mathbb{H})$. For $\nu \in \mathcal{T}(\mathbb{H})$, the sub-sigma algebra of $\mathbb{A}$ generated by real valued $\mathbb{H}$-adapted càdlàg processes stopped at ν will be denoted by $\mathcal{H}_\nu$. The Borel σ-algebra of a Polish space E will be denoted by $\mathcal{B}(E)$.
We introduce some spaces that will play an important role in the sequel. Let $r \geq 1$ and $T \in (0, +\infty]$. $L^r(\mathcal{H}_T, \mathbb{Q})$ (resp. $L^\infty(\mathcal{H}_T, \mathbb{Q})$) is the space of $\mathcal{H}_T$-measurable real valued random variables H such that $\mathbb{E}^{\mathbb{Q}}[|H|^r] < +\infty$ (resp. H is essentially bounded). We denote by $\mathcal{S}_T^r(\mathbb{H}, \mathbb{Q})$ the space of real valued càdlàg $\mathbb{H}$-semimartingales Z such that $||Z||_{\mathcal{S}_T^r(\mathbb{H},\mathbb{Q})} = \mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]} |Z_t|^r\right] < +\infty$.
Equalities between random variables are understood to hold $\mathbb{Q}$-a.s. while equalities between processes are understood to hold $dt \otimes \mathbb{Q}$-a.e. unless mentioned otherwise. Moreover, we will often omit the dependence on $\omega \in \mathbb{A}$ in the notation for random variables and for processes when there is no ambiguity. We will sometimes use the abbreviation r.v. for random variable.

4.2.1 Definitions and assumptions

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. $\mathbb{E}$ denotes the expectation w.r.t. $\mathbb{P}$. Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration of sub-σ-algebras of $\mathcal{A}$. We assume that $\mathcal{F}_0$ is complete and that $\mathbb{F}$ is right-continuous. Let $\tau : \Omega \to \mathbb{R}^+$ be a non-negative random time and D the indicator process of τ defined by

$$D_t \quad = \quad 1_{\{\tau \leq t\}}, \; t \in \mathbb{R}^+.$$

To $\mathbb{F}$ and τ, we associate the filtrations $\mathbb{G}^\tau = (\mathcal{G}_t^\tau)_{t\geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where

$$\mathcal{G}_t^\tau \quad = \quad \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\tau)\right), \; \mathcal{G}_t = \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma\left(D_u, u \leq s\right)\right), t \in \mathbb{R}^+. \tag{4.1}$$

The filtration $\mathbb{F}$ is the reference filtration and represents the information flow that is publicly available to agents, say, in a financial market. The filtration $\mathbb{G}^\tau$ is known as the *initial enlargement* of $\mathbb{F}$ by τ and it is by now a well known ingredient in models for insider trading where τ is considered as the knowledge possessed by some special agent, see [PK96, AIS98, GP97, GP99]. The filtration $\mathbb{G}$ is known as the *progressive enlargement* of $\mathbb{F}$ by τ and it appears in the context of credit risk modeling. Here typically τ represents the default time of some company or assets, see [BR02].
Throughout this work, we will suppose that τ satisfies the so-called *density hypothesis*:

Assumption 4.2.1 (*Density hypothesis*)**.** *For any $t \in \mathbb{R}^+$, the conditional distribution of τ given $\mathcal{F}_t$ is equivalent to the law η of τ, i.e. there exists an $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+) - \mathcal{B}(\mathbb{R}^+)$-measurable map $\alpha_t^d : \Omega \times \mathbb{R}^+ \ni (\omega, u) \mapsto \alpha_t^d(\omega, u) \in (0, +\infty)$ such that*

$$\mathbb{P}\left[\tau \in du | \mathcal{F}_t\right] = \alpha_t^d(\cdot, u)\eta(du) \;\; \mathbb{P}\text{-}a.s.$$

From [Jac85, Lemme 1.8] and [Ame00, Lemma 2.2] we know that under the density hypothesis, there exists an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) - \mathcal{B}(\mathbb{R}+)$-measurable map $\alpha^{\mathrm{d}} : \mathbb{R}^+ \times \Omega \times \mathbb{R}^+ \to (0, +\infty)$ with the properties

- for all $u \in \mathbb{R}^+$, the process $\left(\alpha_t^{\mathrm{d}}(\cdot, u)\right)_{t\geq 0}$ is a càdlàg $(\mathbb{F}, \mathbb{P})$-martingale,
- for any $t \in \mathbb{R}^+$ and for $\mathbb{P}$-almost all $\omega \in \Omega$, the measure $\alpha_t^{\mathrm{d}}(\cdot, u)\mathbb{P}\left(\tau \in du\right)$ on $\left(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+)\right)$ is a version of the conditional distribution $\mathbb{P}\left(\tau \in du | \mathcal{F}_t\right)$.

The process α^{d} is referred to as the *density process* of τ w.r.t. the filtration $\mathbb{F}$ under the measure $\mathbb{P}$.
The *density hypothesis* given by Assumption 4.2.1 is stronger than the standard *Jacod hypothesis* which requires only the absolute continuity of the regular conditional distribution of τ given $\mathcal{F}_t$ w.r.t. to its law η for $t \geq 0$, see [Jac85]. We recall that *Jacod's hypothesis* ensures that an $\mathbb{F}$-semimartingale remains a $\mathbb{G}^\tau$-semimartingale and one can obtain explicitly the corresponding canonical decomposition in the filtration $\mathbb{G}$, see [Jac85, Théorème 2.1]. The preservation of the semimartingale property in passing from $\mathbb{F}$ to $\mathbb{G}^\tau$ is necessary in mathematical finance to investigate for instance the presence of arbitrage opportunities, see [DS94]. The *density hypothesis* in the case where it is satisfied only for $t \in [0, T]$ for some $T \in \mathbb{R}^+$ appears already in [FI93, GP98, Ame00] and all for $t \geq 0$, see [CJZ13]. In addition to the preservation of the semimartingale property between $\mathbb{F} \subseteq \mathbb{G}^\tau$, the density hypothesis preserves as well the *strong predictable representation property* as shown in [Ame00, CJZ13, Fon15].
While the *density hypothesis* is common for the study of the filtration $\mathbb{G}^\tau$, a dominant hypothesis in the study of the filtration $\mathbb{G}$ which leads to the preservation of the semimartingale invariance property under $\mathbb{P}$ is the *immersion* hypothesis or $(\mathcal{H})$-hypothesis, [JLC09a, CJN12, Kus99]:

Assumption 4.2.2. *[$(\mathcal{H})$-hypothesis] Every $(\mathbb{F}, \mathbb{P})$-martingale is a $(\mathbb{G}, \mathbb{P})$-martingale.*

A typical situation in which the immersion hypothesis is satisfied is if τ is independent of $\mathbb{F}$ under $\mathbb{P}$. The immersion hypothesis has the inconvenience of not being stable under a change

of probability measure (see [Kus99]). It turns out to be suitable only for the analysis of before default events and become restrictive as soon as one deals with successive defaults (see [EKJJ10, Remark 4.5]).
Though the *density hypothesis* was initially introduced for the study of the filtration $\mathbb{G}^\tau$ it has been recently adopted for the same purpose for the filtration $\mathbb{G}$, see [JLC09b, EKJJ10, JLC09a, CJZ13]. Observe that $\mathbb{F} \subseteq \mathbb{G} \subseteq \mathbb{G}^\tau$ and it follows from Stricker's theorem [Str77] that the preservation of the semimartingale property holds also between $\mathbb{F}$ and $\mathbb{G}$. The canonical decomposition of an $\mathbb{F}$-local martingale in the filtration $\mathbb{G}$ has been obtained in [JLC09b, CJZ13] and in [CJZ13] it is shown that the *strong predictable representation property* is also preserved between $\mathbb{F}$ and $\mathbb{G}$. The importance of the *density hypothesis* for the filtration $\mathbb{G}$ has been further emphasized by the recent work [EKJJ10] which shows that it is a suitable hypothesis for the analysis of after default events. Moreover, it has the advantage of being stable under change to an equivalent probability measure (see [JLC09a, Proposition 1]) and not being too restrictive when considering the setup of successive defaults, see [EKJJ15a].

Throughout this chapter, we work also under the following standing assumption on η:

Assumption 4.2.3. *η is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^+$, i.e. there exists a positive Borel function $a^{\mathbb{F}}$ such that $\int_0^{+\infty} a^{\mathbb{F}}(u)du = 1$ and $\eta(du) = a^{\mathbb{F}}(u)du$ for every $u \in \mathbb{R}^+$.*

Assumption 4.2.3 entails that the law η of τ is non-atomic. Consequently, τ avoids $\mathbb{F}$-stopping times, i.e. $\mathbb{P}(\tau = \nu) = 0,\ \forall\ \nu \in \mathcal{T}(\mathbb{F})$, see [EKJJ10, Corollary 2.2]. The *conditional survival process* G of τ will play a useful role in the sequel. It is given by

$$G_t \quad = \quad \mathbb{P}\left[\tau > t|\mathcal{F}_t\right] = \int_t^{+\infty} \alpha_t^{\mathrm{d}}(u)\eta(du) = \mathbb{E}\left[\int_t^{\infty} \alpha_u^{\mathrm{d}}(u)\eta(du)\middle|\mathcal{F}_t\right],\ t \in \mathbb{R}^+. \tag{4.2}$$

Since α^{d} is strictly positive and η is non-atomic, G is strictly positive. In [EKJJ10], the authors provide an additive and a multiplicative decomposition of G which we now recall.

Proposition 4.2.4. *Let $\lambda^{\mathbb{F}}$ be the process defined by $\lambda_t^{\mathbb{F}} = \frac{\alpha_t^d(t)}{G_t},\ t \in \mathbb{R}^+$. The Doob-Meyer decomposition of the survival process $(G_t)_{t\geq 0}$ is given by $G_t = 1 + M^{\mathbb{F}} - \int_0^t \alpha_u^d(u)\eta(du), t \geq 0$, where $M^{\mathbb{F}}$ is the square integrable martingale given by*

$$M_t^{\mathbb{F}} \quad = \quad \mathbb{E}\left[\int_0^{\infty} \alpha_u^d(u)\eta(du)\middle|\mathcal{F}_t\right] - 1,\ t \in \mathbb{R}^+.$$

The process G also has a multiplicative decomposition given by $G = L^{\mathbb{F}} e^{-\int_0^{\cdot} \lambda_s^{\mathbb{F}}\eta(ds)}$, where $L^{\mathbb{F}}$ is the $(\mathbb{F}, \mathbb{P})$-local martingale solution of

$$dL_t^{\mathbb{F}} \quad = \quad e^{\int_0^t \lambda_s^{\mathbb{F}}\eta(ds)} dM_t^{\mathbb{F}},\ L_0^{\mathbb{F}} = 1,\ t \geq 0.$$

The process $\lambda^{\mathbb{F}}$ in Proposition 4.2.4 is the $\mathbb{F}$-intensity of τ. For links between the intensity process $\lambda^{\mathbb{F}}$ and the density process α^{d}, we refer to [EKJJ10, Section 4]. The process G appears naturally when computing conditional expectations w.r.t. $\mathcal{G}_t, t \geq 0$ or $\mathcal{G}_\nu,\ \nu \in \mathcal{T}(\mathbb{G})$. We will rely on the processes $L^{\mathbb{F}}$ and $\lambda^{\mathbb{F}}$ to provide equivalent characterizations of $\mathbb{G}$-supermartingales.

4.2.2 Characterization of different measurability properties

In this section we present a description of $\mathbb{G}^\tau$ and $\mathbb{G}$-optional and predictable processes as well as stopping times. We will rely on this description to obtain results regarding the computation of conditional expectations and characterization of supermartingales. The formulas for conditional expectations as well as the characterization of the supermartingales will be essential features for the development of the decomposition approach to address stopping problems in the filtration $\mathbb{G}$ which we present in Section 4.3.

Throughout this section and the subsequent ones, $T \in (0, +\infty)$ is a fixed constant. Let

$$\psi(\omega) \quad = \quad (\omega, \tau(\omega)), \omega \in \Omega.$$

For the characterization of $\mathbb{G}^\tau$-stopping times and the computation of conditional expectations, we introduce the product space $(\widehat{\Omega}, \widehat{\mathcal{A}})$ and the probability measure $\widehat{\mathbb{P}}$ defined as follows

$$\widehat{\Omega} = \Omega \times \mathbb{R}^+, \;\; \widehat{\mathcal{A}} = \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^+) \text{ and } \widehat{\mathbb{P}}(C) = \mathbb{P}\left(\psi^{-1}(C)\right), C \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^+).$$

We consider the filtration $\widehat{\mathbb{F}} = (\widehat{\mathcal{F}}_t)_{t \geq 0}$ where

$$\widehat{\mathcal{F}}_t \quad = \quad \bigcap_{s>t} \left(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}^+)\right), \; t \in \mathbb{R}^+.$$

It will be sometimes convenient to work with the product measure $\mathbb{P} \otimes \eta$. The expectation w.r.t. the measure $\widehat{\mathbb{P}}$ (resp. $\mathbb{P} \otimes \eta$) will be denoted by $\widehat{\mathbb{E}}$ (resp. $\mathbb{E}^{\mathrm{d}}$).

Remark 4.2.5. *We have some useful relationships between the measures $\mathbb{P}, \widehat{\mathbb{P}}$ and $\mathbb{P} \otimes \eta$:*

i) We have $\widehat{\mathbb{P}} \sim \mathbb{P} \otimes \eta$ on $\widehat{\mathcal{F}}_t$ for $t \geq 0$. Indeed, let $s > 0$. For $A \in \mathcal{F}_s$ and $B \in \mathcal{B}(\mathbb{R}^+)$ using the density hypothesis, we obtain that

$$\begin{aligned}
\widehat{\mathbb{P}}(A \times B) \quad &= \quad \mathbb{P}\left(\psi^{-1}(A \times B)\right) \\
&= \quad \mathbb{P}\left(\{\omega \in \Omega : (\omega, \tau(\omega)) \in A \times B\}\right) = \mathbb{P}\left(A \cap \tau^{-1}(B)\right) \\
&= \quad \mathbb{E}\left[1_A 1_B(\tau)\right] = \mathbb{E}\left[\int_0^{+\infty} 1_A 1_B(u) \alpha_s^d(u) \eta(du)\right] = \mathbb{E}^d\left[1_A 1_B \alpha_s^d\right].
\end{aligned}$$

Since A and B are arbitrary, $\widehat{\mathbb{P}} \sim \mathbb{P} \otimes \eta$ on $\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}^+)$ and $d\widehat{\mathbb{P}}/d\mathbb{P} \otimes \eta|_{\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}^+)} = \alpha_s^d$. Now fix $t \geq 0$. Then by definition of $\widehat{\mathcal{F}}_t$, we have $\widehat{\mathcal{F}}_t \subseteq \mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}^+)$ for $s > t$. Hence $\widehat{\mathbb{P}} \sim \mathbb{P} \otimes \eta$ on $\widehat{\mathcal{F}}_t$. We recall that for every $u \in \mathbb{R}^+$ the process $\alpha^d(u)$ is an $(\mathbb{F}, \mathbb{P})$-martingale. Thus α^d is an $\left(\mathcal{F}_s \otimes B(\mathbb{R}^+)\right)_{s \geq 0}$-martingale w.r.t. the measure $\mathbb{P} \otimes \eta$. One verifies that $d\widehat{\mathbb{P}}/d\mathbb{P} \otimes \eta|_{\widehat{\mathcal{F}}_t} = \alpha_t^d$.

Using similar arguments, one shows that for an $\widehat{\mathcal{F}}$-stopping times ν^d satisfying $\nu^d \leq T, \widehat{\mathbb{P}}$-a.s., we have $\widehat{\mathbb{P}} \sim \mathbb{P} \otimes \eta$ on $\widehat{\mathcal{F}}_{\nu^d}$ and $d\widehat{\mathbb{P}}/d\mathbb{P} \otimes \eta|_{\widehat{\mathcal{F}}_{\nu^d}} = \alpha_{\nu^d}^d$.

ii) If $A \in \mathcal{A} \otimes \mathcal{B}(\mathbb{R}^+)$ is a $\widehat{\mathbb{P}}$-null set, then $\psi^{-1}(A)$ is a $\mathbb{P}$-null set. As a result, it follows that for two real valued random variables X^d, Y^d on $\widehat{\Omega}$ such that $X^d = Y^d$ $\widehat{\mathbb{P}}$-a.s., we have $X^d(\tau) = Y^d(\tau)$ $\mathbb{P}$-a.s. A similar result also holds for indistinguishable processes.

Notation: For $X^{\mathrm{d}} : \widehat{\Omega} \to \mathbb{R}$, $u \in \mathbb{R}^+$, we denote by $X^{\mathrm{d}}(u)$ (resp. $X^{\mathrm{d}}(\tau)$) the map $\Omega \ni \omega \mapsto X(\omega, u)$ (resp. $X^{\mathrm{d}}(\omega, \tau(\omega))$). Similarly for a process $X^{\mathrm{d}} : \mathbb{R}^+ \times \Omega \times \mathbb{R}^+ \to \mathbb{R}$ and $u \in \mathbb{R}^+$, we denote by $X^{\mathrm{d}}(u)$ (resp. $X^{\mathrm{d}}(\tau)$) the map $\mathbb{R}^+ \times \Omega \ni (t, \omega) \mapsto X_t^{\mathrm{d}}(\omega, u)$ (resp. $X_t^{\mathrm{d}}(\omega, \tau(\omega))$). We use a similar notation for processes defined on $[0, T]$.

We will now give the different measurability properties. We begin with the following lemma which gives a full characterization of $\mathbb{G}^\tau$-predictable processes and $\mathbb{G}^\tau$-stopping times.

Proposition 4.2.6. *The following hold:*

i) A mapping $Y : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is $\mathcal{P}(\mathbb{G}^\tau)$-measurable if and only if there exists a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable $Y^d : \mathbb{R}^+ \times \widehat{\Omega} \to \mathbb{R}$ such that $Y = Y^d(\tau)$.

ii) Let $\gamma : \Omega \to \mathbb{R}^+$ be a random time. The random time γ is a $\mathbb{G}^\tau$-stopping time if and only if there exists an $\widehat{\mathbb{F}}$-stopping time $\gamma^d : \widehat{\Omega} \to \mathbb{R}^+$ satisfying $\gamma = \gamma^d(\tau)$. If γ is such that $\gamma \le T$ $\mathbb{P}$-a.s., then γ^d can be chosen such that $\gamma^d \le T$ $\widehat{\mathbb{P}}$-a.s..

Proof. The proof of assertion i) can be found e.g. in [Jeu80, CJZ13]. Assertion ii) was recently obtained in [EI18, Proposition 3.4]. □

Remark 4.2.7. *The following observations will be quite useful.*

i) Let $\widehat{\mathbb{F}}^{\widehat{\mathbb{P}}}$ be the completion of $\widehat{\mathbb{F}}$ w.r.t. $\widehat{\mathbb{P}}$-null sets. By Lemma I.1.19 in [JS03], for every $\gamma^d \in \mathcal{T}(\widehat{\mathbb{F}}^{\widehat{\mathbb{P}}})$, there exists $\gamma^{d'} \in \mathcal{T}(\widehat{\mathbb{F}})$ such that $\gamma^d = \gamma^{d'}$ $\widehat{\mathbb{P}}$-a.s.. Due to Remark 4.2.5 ii), we can therefore replace $\widehat{\mathbb{F}}$ in Proposition 4.2.6 by $\widehat{\mathbb{F}}^{\widehat{\mathbb{P}}}$.

ii) The set $\mathcal{T}(\mathbb{F})$ is naturally embedded in $\mathcal{T}(\widehat{\mathbb{F}})$. Indeed for $\gamma \in \mathcal{T}(\mathbb{F})$, one has $\gamma = \gamma^d(\tau)$, where $\gamma^d(\omega, u) = \gamma(\omega), (\omega, u) \in \widehat{\Omega}$. Note that $\tau = \tau^d(\tau)$ where $\tau^d(\omega, u) = u, \ \forall\ (\omega, u) \in \widehat{\Omega}$.

iii) Let $\nu^d \in \mathcal{T}(\widehat{\mathbb{F}})$, then $\nu^d(u) \in \mathcal{T}(\mathbb{F})$ for every $u \in \mathbb{R}^+$. Indeed, Let $u \in \mathbb{R}^+$ and $t \in \mathbb{R}^+$. Define $A^u = \left\{\nu^d(u) \le t\right\}$ and $A = \left\{\nu^d \le t\right\}$. A^u is the u-section of A in Ω. Since $\nu^d \in \mathcal{T}(\widehat{\mathbb{F}})$, $A \in \widehat{\mathcal{F}}_t = \bigcap_{s>t} \left(\mathcal{F}_s \otimes \mathcal{B}(\mathbb{R}^+)\right)$ and therefore $A^u \in \mathcal{F}_s, \forall s > t$. $\mathbb{F}$ being right-continuous, $A^u \in \mathcal{F}_t$. As t is arbitrary, $\nu^d(u) \in \mathcal{T}(\mathbb{F})$.

The following proposition gives a characterization of $\mathbb{G}^\tau$-optional processes.

Proposition 4.2.8. *Let $X : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ be a $\mathcal{B}(\mathbb{R}^+) \otimes \mathcal{A} - \mathcal{B}(\mathbb{R})$-measurable map and $\nu^d \in \mathcal{T}(\widehat{\mathbb{F}})$ with $\widehat{\mathbb{P}}\left(\nu^d < +\infty\right) = 1$. Then:*

a) X is $\mathcal{O}(\mathbb{G}^\tau)$-measurable if and only there exists an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) - \mathcal{B}(\mathbb{R})$-measurable map $X^d : \mathbb{R}^+ \times \widehat{\Omega} \to \mathbb{R}$ such that $X = X^d(\tau)$.

b) ξ is $\mathcal{G}^\tau_{\nu^d(\tau)}$-measurable if and only if there exists an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process ξ^d such that $\xi = \xi^d_{\nu^d(\tau)}(\tau)$.

c) $\mathcal{G}^\tau_\tau = \mathcal{F}_\tau$.

Proof. a) We refer to [Fon15, Lemma 4.3].

b) By [HWY92, Corollary 3.23] ξ is $\mathcal{G}^\tau_{\nu^{\mathrm{d}}(\tau)}$-measurable if and only if there exists an $\mathcal{O}(\mathbb{G}^\tau)$-measurable process X such that $\xi = X_{\nu^{\mathrm{d}}(\tau)}$. The characterization of $\mathcal{O}(\mathbb{G}^\tau)$ obtained in a) gives the result.

c) Clearly $\mathcal{F}_\tau \subseteq \mathcal{G}^\tau_\tau$. By the monotone class theorem, for an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) - \mathcal{B}(\mathbb{R})$-measurable process ξ^{d}, $\xi^{\mathrm{d}}_\tau(\tau)$ is $\mathcal{F}_\tau$-measurable . It follows from b) that $\mathcal{G}^\tau_\tau \subseteq \mathcal{F}_\tau$. The proof is complete. □

We now turn our attention to the filtration $\mathbb{G}$. It is well known that for every $t \ge 0, \mathcal{F}_t \cap \{t > \tau\} = \mathcal{G}_t \cap \{\tau > t\}$. This translates the fact that $\mathbb{G}$ contains the same information as $\mathbb{F}$ before the time τ. Observe that $\mathbb{G} \subseteq \mathbb{G}^\tau$ and thus some insights on the filtration $\mathbb{G}$ can be gained from $\mathbb{G}^\tau$. In the recent work [KLP13], the authors link the filtration $\mathbb{G}^\tau$ and $\mathbb{G}$ to provide a detailed study of $\mathbb{G}$. The link between $\mathbb{G}^\tau$ and $\mathbb{G}$ is achieved by using a suitable notion of what it means for two filtrations to coincide after a random time. The precise concept of this notion is given by the following.

Definition 4.2.9. *Let $\mathbb{H}^1, \mathbb{H}^2$ be two filtrations such that $\mathbb{H}^1 \subseteq \mathbb{H}^2$, and let γ be an $\mathbb{H}^2$-stopping time. Then $\mathbb{H}^1$ and $\mathbb{H}^2$ are said to coincide after γ if for every $\mathbb{H}^2$-adapted process X, the process $1_{[\gamma,\infty[}(X - X_\gamma)$ is $\mathbb{H}^1$-adapted.*

The following lemma asserts that $\mathbb{G}$ and $\mathbb{G}^\tau$ coincide after τ. This property will be used to obtain a full characterization of $\mathbb{G}$-stopping times.

Lemma 4.2.10. *[KLP13, Lemmas 2 and 3] The following relations between $\mathbb{G}$ and $\mathbb{G}^\tau$ hold:*

i) The filtrations $\mathbb{G}$ and $\mathbb{G}^\tau$ coincide after τ.

ii) For every $\mathbb{G}^\tau$ stopping time $\gamma, \gamma \vee \tau$ is a $\mathbb{G}$-stopping time.

The following lemma gives the representation of $\mathbb{G}$-predictable and optional processes in terms of $\mathbb{F}$-predictable and optional processes. The characterization of $\mathbb{G}$-predictable processes relies on a monotone class type argument [Jeu79, CJZ13] while the case of optional processes requires a deeper analysis developed in [Son14].

Lemma 4.2.11. *The following assertions hold:*

1. *A mapping $K : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is $\mathcal{P}(\mathbb{G})$-measurable if and only if there exists a $\mathcal{P}(\mathbb{F})$-measurable random variable K^b and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable K^d such that*

$$K_t = K_t^b 1_{\{t \le \tau\}} + K_t^d(\tau) 1_{\{t > \tau\}}, \ t \in \mathbb{R}^+. \tag{4.3}$$

2. *A mapping $K : \mathbb{R}^+ \times \Omega \to \mathbb{R}$ is $\mathcal{O}(\mathbb{G})$-measurable if and only if there exists an $\mathcal{O}(\mathbb{F})$-measurable random variable K^b and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable K^d such that*

$$K_t = K_t^b 1_{\{t < \tau\}} + K_t^d(\tau) 1_{\{t \ge \tau\}}, \ t \in \mathbb{R}^+. \tag{4.4}$$

The decomposition of $\mathbb{G}$-optional processes given by (4.4) is known as the *optional splitting formula*, see [Son14]. We will use the following terminology. For a $\mathbb{G}$-optional process K satisfying (4.4), we call K^{b} (resp. $K^{\mathrm{d}}(\tau)$) the *pre-default* value (resp. post-default) of K. We will employ a similar terminology for a $\mathbb{G}$-predictable process K satisfying (4.3).

A splitting feature for $\mathbb{G}$-stopping times similar to that of $\mathbb{G}$-optional processes has been obtained[1]in [BZ14, Theorem 2.1].

Proposition 4.2.12. *Let $\gamma : \Omega \to \mathbb{R}^+$. Then γ is a $\mathbb{G}$-stopping time if and only if there exists $\gamma^b \in \mathcal{T}(\mathbb{F})$ and $\gamma^d \in \mathcal{T}(\widehat{\mathbb{F}})$ such that*

$$\gamma = \gamma^b 1_{\{\gamma^b < \tau\}} + \gamma^d(\tau) 1_{\{\gamma^b \ge \tau\}}, \tag{4.5}$$

$$\gamma^d(\tau) \ge \tau. \tag{4.6}$$

Moreover if $\gamma \le T$, then γ^b can be replaced by $\gamma^b \wedge T$ and $\left\{\gamma^d(\tau) \le T\right\} = \{\tau \le T\}$.

We propose here a shorter and alternative proof[2] [BZ14, Theorem 2.1].

Proof. " $\Leftarrow$" . First we show that every random variable γ that satisfies (4.5) and (4.6) is a $\mathbb{G}$-stopping time, i.e $\gamma \in \mathcal{T}(\mathbb{G})$. Let $t \in \mathbb{R}^+$. The sets $\left\{\gamma^{\mathrm{b}} < \tau\right\}$ and $\left\{\gamma^{\mathrm{b}} \ge \tau\right\}$ being disjoint, we have

$$\{\gamma \le t\} = \left(\left\{\gamma^{\mathrm{b}} \le t\right\} \cap \left\{\gamma^{\mathrm{b}} < \tau\right\}\right) \bigcup \left(\left\{\gamma^{\mathrm{d}}(\tau) \le t\right\} \cap \left\{\gamma^{\mathrm{b}} \ge \tau\right\}\right) = A \bigcup B,$$

where $A = \left\{\gamma^{\mathrm{b}} \le t\right\} \cap \left\{\gamma^{\mathrm{b}} < \tau\right\}$ and $B = \left\{\gamma^{\mathrm{d}}(\tau) \le t\right\} \cap \left\{\gamma^{\mathrm{b}} \ge \tau\right\}$. Since $\gamma^{\mathrm{b}} \in \mathcal{T}(\mathbb{F}) \subseteq \mathcal{T}(\mathbb{G})$, $A \in \mathcal{G}_t$. Due to (4.6), $\left\{\gamma^{\mathrm{d}}(\tau) \le t\right\} = \left\{\gamma^{\mathrm{d}}(\tau) \le t\right\} \cap \{\tau \le t\}$ and Lemma 4.2.10 implies

[1] The result in [BZ14] is obtained in the setting where the filtration $\mathbb{F}$ is progressively enlarged with multiple random times $\tau_1, \cdots, \tau_n, n \in \mathbb{N}$

[2] The proof can be adapted to the setting of multiple default times in [BZ14] using the corresponding optional splitting formula for $\mathbb{G}$-optional processes given in [Son14, Theorem 6.5].

that $\gamma^{\mathrm{d}}(\tau) \in \mathcal{T}(\mathbb{G})$. We therefore have $B = \left(\left\{\gamma^{\mathrm{d}}(\tau) \leq t\right\} \cap \{\tau \leq t\}\right) \bigcap \left\{\tau \leq \gamma^{\mathrm{b}} \wedge t\right\} \in \mathcal{G}_t$ and $\{\gamma \leq t\} \in \mathcal{G}_t$. Hence $\gamma \in \mathcal{T}(\mathbb{G})$.
$'' \Rightarrow''$. Let H be the process defined by $H_t = 1_{[0,\gamma)}(t), t \geq 0$. Clearly H is $\mathcal{O}(\mathbb{G})$-measurable and by Lemma 4.2.11 there exists an $\mathcal{O}(\mathbb{F})$-measurable function H^{b} and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable function H^{d} such that

$$H_t \quad = \quad H^{\mathrm{b}}_t 1_{\{t<\tau\}} + H^{\mathrm{d}}_t(\tau) 1_{\{t \geq \tau\}}, t \geq 0.$$

Let $\gamma^{\mathrm{b}} := \inf \left\{t \geq 0, H^{\mathrm{b}}_t = 0\right\}$. Then $\gamma^{\mathrm{b}} \in \mathcal{T}(\mathbb{F})$ as H^{b} is $\mathcal{O}(\mathbb{F})$-measurable. From the equality $1_{[0,\gamma\wedge\tau)}(t) = H^{\mathrm{b}} 1_{[0,\tau)}, t \geq 0$, and the definition of γ^{b}, we have $\gamma^{\mathrm{b}} \wedge \tau = \gamma \wedge \tau$. Thus γ has the decomposition

$$\gamma \quad = \quad \gamma 1_{\{\gamma^{\mathrm{b}}<\tau\}} + \gamma 1_{\{\gamma^{\mathrm{b}} \geq \tau\}} = \gamma^{\mathrm{b}} 1_{\{\gamma^{\mathrm{b}}<\tau\}} + \gamma 1_{\{\gamma^{\mathrm{b}} \geq \tau\}}.$$

On $\{\gamma^{\mathrm{b}} \geq \tau\}$, we have $\gamma \geq \tau$ since $\gamma^{\mathrm{b}} \wedge \tau = \gamma \wedge \tau$. Consequently on the event $\{\gamma^{\mathrm{b}} \geq \tau\}$, $H = 1$ on $[0, \tau)$. It follows that on $\{\gamma^{\mathrm{b}} \geq \tau\}$, we have

$$\begin{aligned}
\gamma \quad &= \quad \inf\{t \in \mathbb{R}^+, H_t = 0\} = \inf\{t \geq \tau, H_t = 0\} \\
&= \quad \inf\{t \geq \tau, H^{\mathrm{d}}_t(\tau) = 0\} = \inf\{t \in \mathbb{R}^+, H^{\mathrm{d}}_t(\tau) 1_{[\tau,+\infty[} - 1_{[0,\tau)}(t) = 0\} = \gamma^{\mathrm{d}}(\tau)
\end{aligned}$$

where

$$\gamma^{\mathrm{d}}(\omega, u) \quad = \quad \inf\{t \geq 0, H^{\mathrm{d}}_t(\omega, u) 1_{[u,\infty[} - 1_{[0,u)} = 0\}, (\omega, u) \in \Omega \times \mathbb{R}^+.$$

Clearly $J = H^{\mathrm{d}}(\tau) 1_{[\tau,+\infty[} - 1_{[0,\tau[}$ is $\mathcal{O}(\mathbb{G}^\tau)$-measurable. Hence $\gamma^{\mathrm{d}}(\tau) \in \mathcal{T}(\mathbb{G}^\tau)$. Moreover by construction of J, we have $\gamma^{\mathrm{d}}(\tau) \geq \tau$. With γ^{b} and $\gamma^{\mathrm{d}}(\tau)$ as constructed above, (4.5) and (4.6) are satisfied.

The last statement is obtained by replacing γ with $\gamma \wedge T$ and using the fact that $\gamma = \gamma^{\mathrm{b}}$ on $\left\{\gamma^{\mathrm{b}} < \tau\right\}$. □

In analogy to the optional splitting formula for $\mathbb{G}$-optional processes, for $\gamma \in \mathcal{T}(\mathbb{G})$, we call the representation (4.5) the optional splitting formula for γ and γ^{b} (resp. $\gamma^{\mathrm{d}}(\tau)$) the pre-default (resp. post-default) value of γ. For $\gamma^{\mathrm{d}} \in \mathcal{T}(\widehat{\mathbb{F}})$, we denote by γ^{d}_T the $\widehat{\mathbb{F}}$-stopping time $\gamma^{\mathrm{d}} \wedge T$.

Remark 4.2.13. *The optional splitting formula of γ has a simple structure in some particular cases:*

i) For the particular case $\gamma = \tau$, we take $\gamma^b = +\infty$ and $\gamma^d(\omega, u) = \tau^d(\omega, u) = u$, $\forall\ (\omega, u) \in \widehat{\Omega}$. For an $\mathbb{F}$-stopping time γ, we have $\gamma^b = \gamma$ and $\gamma^d(\tau) = \gamma \vee \tau$. Finally, for a $\mathbb{G}^\tau$-stopping times γ bigger than $\tau, \gamma^b = +\infty$.

ii) By Proposition 4.2.12, $\gamma \in \mathcal{T}_\tau(\mathbb{G})$ if and only if $\gamma = \gamma^b 1_{\{\gamma^b<\tau\}} + \tau 1_{\{\gamma^b \geq \tau\}}$ for some $\gamma^b \in \mathcal{T}(\mathbb{F})$. In particular every $\mathbb{G}$-stopping time strictly less than τ is in fact an $\mathbb{F}$-stopping time.

iii) As $T \wedge \tau \leq \tau$, ii) implies that $\gamma \in \mathcal{T}_{T\wedge\tau}(\mathbb{G})$ if and only if $\gamma = \gamma^b 1_{\{\gamma^b<\tau\}} + \tau 1_{\{\gamma^b \geq \tau\}}, \gamma^b \in \mathcal{T}_T(\mathbb{F})$.

Let $\nu, \gamma \in \mathcal{T}_T(\mathbb{G})$ and $\nu^{\mathrm{b}}, \gamma^{\mathrm{b}} \in \mathcal{T}(\mathbb{F}), \nu^{\mathrm{d}}, \gamma^{\mathrm{d}} \in \mathcal{T}(\widehat{\mathbb{F}})$ such that

$$\nu = \nu^{\mathrm{b}} 1_{\{\nu^{\mathrm{b}}<\tau\}} + \nu^{\mathrm{d}}(\tau) 1_{\{\nu^{\mathrm{b}} \geq \tau\}} \text{ and } \gamma = \gamma^{\mathrm{b}} 1_{\{\gamma^{\mathrm{b}}<\tau\}} + \gamma^{\mathrm{d}}(\tau) 1_{\{\gamma^{\mathrm{b}} \geq \tau\}}. \tag{4.7}$$

Now assume that $\gamma \in \mathcal{T}_{\nu,T}(\mathbb{G})$. Then $\gamma^{\mathrm{b}} \geq \nu^{\mathrm{b}}$ on the event $\{\nu^{\mathrm{b}} < \tau\}$. However on the event $\{\nu^{\mathrm{b}} \geq \tau\}$, ν and γ take precisely the values $\nu^{\mathrm{d}}(\tau)$ and $\gamma^{\mathrm{d}}(\tau)$. This hinders a justification for the inequality $\gamma^{\mathrm{b}} \geq \nu^{\mathrm{b}}$. In the following lemma, we give an alternative representation of γ for which its pre-default value $\widetilde{\gamma}^{\mathrm{b}}$ satisfies $\widetilde{\gamma}^{\mathrm{b}} \geq \nu^{\mathrm{b}}$. The latter representation will be useful in Proposition 4.2.18 for the computation of conditional expectations.

Lemma 4.2.14. *Let $\nu, \gamma \in \mathcal{T}(\mathbb{G})$ with optional splitting formulas given by (4.7).*

i) If $\gamma \in \mathcal{T}_{\nu,T}(\mathbb{G})$, then there exists $\delta \in \mathcal{T}_{\nu^d_T(\tau),T}(\mathbb{G}^\tau)$ such that $\gamma = \delta$ on $\left\{\nu^b \geq \tau\right\}$. Moreover,

$$\gamma \quad = \quad (\gamma^b \vee \nu^b)1_{\{\gamma^b \vee \nu^b < \tau\}} + \gamma^d(\tau)1_{\{\gamma^b \vee \nu^b \geq \tau\}}. \tag{4.8}$$

If $(\gamma^b, \gamma^d(\tau)) \in \mathcal{T}_{\nu^b,T}(\mathbb{F}) \times \mathcal{T}_{\nu^d_T(\tau),T}(\mathbb{G}^\tau)$, then γ given by (4.8) satisfies $\gamma \in \mathcal{T}_{\nu,T}(\mathbb{G})$.

Proof. Assume that $\gamma \in \mathcal{T}_{\nu,T}(\mathbb{G})$. Let $\bar{\gamma}^{\mathrm{d}}(\tau) = \nu^{\mathrm{d}}_T(\tau)1_{\{\nu^{\mathrm{b}}<\tau\}} + \gamma^{\mathrm{d}}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}$. As $\gamma^{\mathrm{d}}(\tau) \geq \nu^{\mathrm{d}}(\tau)$ on $\left\{\nu^{\mathrm{b}} \geq \tau\right\}$, $\bar{\gamma}^{\mathrm{d}}(\tau) \in \mathcal{T}_{\nu^{\mathrm{d}}_T(\tau),T}(\mathbb{G}^\tau)$. Clearly $\gamma = \gamma^{\mathrm{d}}(\tau) = \bar{\gamma}^{\mathrm{d}}(\tau)$ on $\left\{\nu^{\mathrm{b}} \geq \tau\right\}$. We take $\delta = \bar{\gamma}^{\mathrm{d}}(\tau)$. To show (4.8), we note that $\gamma = \gamma \vee \nu = (\gamma \vee \nu)1_{\{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}<\tau\}} + (\gamma \vee \nu)1_{\{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\geq\tau\}}$. On the event $\{\gamma^{\mathrm{b}} \vee \nu^{\mathrm{b}} < \tau\}, \gamma = \gamma^{\mathrm{b}}$ and $\nu = \nu^{\mathrm{b}} \leq \gamma^{\mathrm{b}}$. Clearly $\nu = \nu^{\mathrm{d}}(\tau) \geq \tau$ implies that $\gamma = \gamma^{\mathrm{d}}(\tau)$. Therefore $\{\nu^{\mathrm{b}} \geq \tau\} \subseteq \{\gamma^{\mathrm{b}} \geq \tau\}$ and $\nu \vee \gamma = \gamma^{\mathrm{d}}(\tau)$ on the event $\{\gamma^{\mathrm{b}} \vee \nu^{\mathrm{b}} \geq \tau\}$. Thus (4.8) holds. Assume that $(\gamma^{\mathrm{b}}, \gamma^{\mathrm{d}}(\tau)) \in \mathcal{T}_{\nu^{\mathrm{b}},T}(\mathbb{F}) \times \mathcal{T}_{\nu^{\mathrm{d}}_T(\tau),T}(\mathbb{G}^\tau)$. Since $\gamma^{\mathrm{b}} \leq T$ and $\nu^{\mathrm{d}}(\tau) \geq \tau$, on the event $\{\gamma^{\mathrm{b}} \geq \tau\}$ one has $\gamma^{\mathrm{d}}(\tau) \geq \nu^{\mathrm{d}}_T(\tau) = \nu^{\mathrm{d}}(\tau) \wedge T \geq \tau$. Hence $\gamma = \gamma^{\mathrm{b}}1_{\{\gamma^{\mathrm{b}}<\tau\}}+(\gamma^{\mathrm{d}}(\tau) \vee \tau)1_{\{\gamma^{\mathrm{b}}\geq\tau\}}$ and Proposition 4.2.12 implies that $\gamma \in \mathcal{T}_T(\mathbb{G})$. Due to the hypothesis on γ^{b} and $\gamma^{\mathrm{d}}(\tau)$, $\gamma \in \mathcal{T}_{\nu,T}(\mathbb{G})$. □

The following result which gives the characterization of the σ-algebra $\mathcal{G}_\gamma$ for $\gamma \in \mathcal{T}(\mathbb{G})$ is a corollary of Proposition 4.2.12.

Corollary 4.2.15. *Let $\gamma \in \mathcal{T}(\mathbb{G})$ with $\mathbb{P}(\gamma < +\infty) = 1$. Let $\gamma^b \in \mathcal{T}(\mathbb{F})$ and $\gamma^d \in \mathcal{T}(\widehat{\mathbb{F}})$ such that $\gamma = \gamma^b 1_{\{\gamma^b<\tau\}} + \gamma^d(\tau)1_{\{\gamma^b\geq\tau\}}$. Let ξ be a positive mapping defined on Ω. ξ is $\mathcal{G}_\gamma$-measurable if and only if $\xi = \xi^b 1_{\{\gamma^b<\tau\}} + \xi^d(\tau)1_{\{\gamma^b\geq\tau\}}$ where ξ^b is an $\mathcal{F}_{\gamma^b}$-measurable random variable and $\xi^d(\tau)$ a $\mathcal{G}^\tau_{\gamma^d(\tau)}$-measurable random variable. Moreover, $\mathcal{G}^\tau_\tau = \mathcal{G}_\tau = \mathcal{F}_\tau$.*

Proof. $'' \Rightarrow''$. Let ξ be $\mathcal{G}_\gamma$-measurable. By [HWY92, Corollary 3.23], there exists an $\mathcal{O}(\mathbb{G})$-measurable random variable X such that $\xi = X_\gamma$. We infer from Lemma 4.2.11 that there exists an $\mathcal{O}(\mathbb{F})$-measurable random variable X^{b} and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable X^{d} such that

$$X_t = X^{\mathrm{b}}_t 1_{\{t<\tau\}} + X^{\mathrm{d}}_t(\tau)1_{\{t\geq\tau\}}, t \geq 0.$$

Due to the optional splitting formula for γ, one obtains that

$$\xi = X_\gamma = X^{\mathrm{b}}_{\gamma^{\mathrm{b}}}1_{\{\gamma^{\mathrm{b}}<\tau\}} + X^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)1_{\{\gamma^{\mathrm{b}}\geq\tau\}}.$$

Thus to obtain the splitting formula for ξ, we take $\xi^{\mathrm{b}} = X^{\mathrm{b}}_{\gamma^{\mathrm{b}}}$ and $\xi^{\mathrm{d}}(\tau) = X^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)$.
$'' \Leftarrow''$. Let $\xi = \xi^{\mathrm{b}}1_{\{\gamma^{\mathrm{b}}<\tau\}} + \xi^{\mathrm{d}}(\tau)1_{\{\gamma^{\mathrm{b}}\geq\tau\}}$ where ξ^{b} is $\mathcal{F}_{\gamma^{\mathrm{b}}}$-measurable and $\xi^{\mathrm{d}}(\tau)$ is $\mathcal{G}^\tau_{\gamma^{\mathrm{d}}(\tau)}$-measurable. We want to show that ξ is $\mathcal{G}_\gamma$-measurable. Let $A \in \mathcal{B}(\mathbb{R})$ and $t \in \mathbb{R}^+$. Using the fact that $\{\gamma^{\mathrm{b}} < \tau\}$ and $\{\gamma^{\mathrm{b}} \geq \tau\}$ are disjoint, we can write $K = \{\xi \in A\} \cap \{\gamma \leq t\}$ as follows

$$K = \quad \left(\{\xi^{\mathrm{b}} \in A\} \cap \{\gamma^{\mathrm{b}} < \tau\} \cap \{\gamma^{\mathrm{b}} \leq t\}\right) \bigcup \left(\{\xi^{\mathrm{d}}(\tau) \in A\} \cap \{\gamma^{\mathrm{b}} \geq \tau\} \cap \{\gamma^{\mathrm{d}}(\tau) \leq t\}\right). \tag{4.9}$$

Clearly $\{\xi^{\mathrm{b}} \in A\} \cap \{\gamma^{\mathrm{b}} < \tau\} \cap \{\gamma^{\mathrm{b}} \leq t\} = \{\xi^{\mathrm{b}} \in A\} \cap \{\gamma^{\mathrm{b}} \leq t\} \bigcap \{\gamma^{\mathrm{b}} < \tau\} \cap \{\gamma^{\mathrm{b}} \leq t\} \in \mathcal{G}_t$. The inequality $\gamma^{\mathrm{d}}(\tau) \geq \tau$ yields

$$\{\xi^{\mathrm{d}}(\tau) \in A\} \cap \{\gamma^{\mathrm{b}} \geq \tau\} \cap \{\gamma^{\mathrm{d}}(\tau) \leq t\} = \{\xi^{\mathrm{d}}(\tau) \in A\} \cap \{\gamma^{\mathrm{d}}(\tau) \leq t\} \bigcap \{\tau \leq \gamma^{\mathrm{d}} \wedge t\} \in \mathcal{G}_t.$$

Hence $\{\xi \in A\} \cap \{\gamma \leq t\} \in \mathcal{G}_t$. Since A and t are arbitrary, we infer that ξ is $\mathcal{G}_\gamma$-measurable. Clearly $\mathcal{F}_\tau \subseteq \mathcal{G}_\tau \subseteq \mathcal{G}^\tau_\tau$. By Proposition 4.2.8 $\mathcal{F}_\tau = \mathcal{G}^\tau_\tau$. We deduce that $\mathcal{F}_\tau = \mathcal{G}_\tau = \mathcal{G}^\tau_\tau$. □

4.2.3 Computation of conditional expectations

In the theory of filtration enlargements, the computation of conditional expectations is a crucial tool. The decomposition of $\mathbb{F}$-local martingales in the filtration $\mathbb{G}^\tau$ and $\mathbb{G}$ relies on this. The focus in this section is to extend results from [CJZ13, EKJJ10, EKJJ15b] on conditional expectations w.r.t. $\mathcal{H}_t \in \{\mathcal{G}_t^\tau, \mathcal{G}_t\}, t \in \mathbb{R}^+$ to σ-algebras $\mathcal{G}_\nu^\tau, \nu \in \mathcal{T}_T(\mathbb{G}^\tau)$ or $\mathcal{G}_\nu, \nu \in \mathcal{T}_T(\mathbb{G})$.
First we provide some properties that will be often used, and the link between the spaces $L^1(\Omega, \mathcal{G}_T^\tau, \mathbb{P})$ and $L^1(\widehat{\Omega}, \widehat{\mathcal{F}}_T, \widehat{\mathbb{P}})$.

Lemma 4.2.16. *Let $r \geq 1$ and Y^d an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process.*

i) For every $t \in [0,T]$, $Y_t^d \in L^r\left(\widehat{\Omega}, \widehat{\mathcal{F}}_t, \widehat{\mathbb{P}}\right)$ if and only if $Y_t^d(\tau) \in L^r\left(\Omega, \mathcal{G}_t^\tau, \mathbb{P}\right)$.

ii) Let $\nu^d \in \mathcal{T}(\widehat{\mathbb{F}})$ such that $Y_{\nu^d(\tau)}^d(\tau) 1_{\{\nu^d(\tau) \leq T\}} \leq 0$ $\mathbb{P}$-a.s., and belongs to $L^1(\Omega, \mathcal{G}_T^\tau, \mathbb{P})$. Then $Y_{\nu^d(u)}^d(u) \leq 0$ $\mathbb{P}$-a.s. for η-almost all $u \leq T$.

Proof. i) Let $r \geq 1$ and $t \in [0,T]$. The assertion follows from the equalities

$$\mathbb{E}\left[|Y_t^{\mathrm{d}}(\tau)|^r\right] = \mathbb{E}\left[\int_0^\infty |Y_t^{\mathrm{d}}(u)|^r \alpha_t^{\mathrm{d}}(u)\eta(du)\right] = \mathbb{E}^{\mathrm{d}}\left[|Y_t^{\mathrm{d}}|^r \alpha_t^{\mathrm{d}}\right] = \widehat{\mathbb{E}}\left[|Y_t^{\mathrm{d}}|^r\right].$$

ii) By hypothesis, $\max\{Y_{\nu^{\mathrm{d}}(\tau)}^{\mathrm{d}}(\tau) 1_{\{\nu^{\mathrm{d}}(\tau) \leq T\}}, 0\} = 0$. Using the density hypothesis and the martingale property of $\alpha^{\mathrm{d}}(u)$ for every $u \geq 0$, we deduce that

$$\begin{aligned}
\mathbb{E}\left[\max\{Y_{\nu^{\mathrm{d}}(\tau)}^{\mathrm{d}}(\tau) 1_{\{\nu^{\mathrm{d}}(\tau) \leq T\}}, 0\}\right] &= \mathbb{E}\left[\max\{Y_{\nu^{\mathrm{d}}(\tau)}^{\mathrm{d}}(\tau), 0\} 1_{\{\nu^{\mathrm{d}}(\tau) \leq T\}}\right] \\
&= \mathbb{E}\left[\int_0^T \max\{Y_{\nu^{\mathrm{d}}(u)}^{\mathrm{d}}(u), 0\} \alpha_{\nu^{\mathrm{d}}(u)}^{\mathrm{d}}(u)\eta(du)\right] = 0.
\end{aligned}$$

The process α^{d} being strictly positive, we infer from Fubini's theorem that $Y_{\nu^{\mathrm{d}}(u)}^{\mathrm{d}}(u) \leq 0$ $\mathbb{P}$-a.s. for η-almost all $u \in [0,T]$. $\square$

Next we show how to compute conditional expectations w.r.t. $\mathcal{G}_\nu^\tau$ for $\nu \in \mathcal{T}_T(\mathbb{G}^\tau)$.

Proposition 4.2.17. *Let $\nu^d, \gamma^d \in \mathcal{T}_T(\widehat{\mathbb{F}})$ with $\nu^d \leq \gamma^d$ $\widehat{\mathbb{P}}$-a.s. and $\nu = \nu^d(\tau), \gamma = \gamma^d(\tau)$. Let ξ^d be an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable such that $\xi = \xi_{\gamma^d(\tau)}^d(\tau) \in L^1(\Omega, \mathcal{G}_\gamma^\tau, \mathbb{P})$. We have the equality*

$$\mathbb{E}\left[\xi | \mathcal{G}_\nu^\tau\right] = \frac{\mathbb{E}^d\left[\xi_{\gamma^d}^d \alpha_{\gamma^d}^d | \widehat{\mathcal{F}}_{\nu^d}\right](\tau)}{\alpha_\nu^d(\tau)} = \widehat{\mathbb{E}}\left[\xi_{\gamma^d}^d | \widehat{\mathcal{F}}_{\nu^d}\right](\tau). \tag{4.10}$$

If additionally $\nu \in \mathcal{T}_T(\mathbb{F})$, then

$$\mathbb{E}\left[\xi | \mathcal{G}_\nu^\tau\right] = \frac{\mathbb{E}\left[\xi_{\gamma^d(u)}^d(u) \alpha_{\gamma^d(u)}^d(u) | \mathcal{F}_\nu\right]|_{u=\tau}}{\alpha_\nu^d(\tau)}. \tag{4.11}$$

Proof. The expressions on the right hand side of (4.10) and (4.11) are well defined by Lemma 4.2.16 since $\xi \in L^1(\Omega, \mathcal{G}_\gamma^\tau, \mathbb{P})$. We begin with the proof of (4.10). Let Z^{d} be a bounded $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable. By the law of iterated conditional expectation and the density hypothesis, we have

$$\begin{aligned}
\mathbb{E}\left[\xi Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)\right] &= \mathbb{E}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau) Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)\right] = \mathbb{E}\left[\mathbb{E}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau) Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)|\mathcal{F}_T\right]\right] \\
&= \mathbb{E}\left[\int_0^\infty \xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u) Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_T(u)\eta(du)\right] = \mathbb{E}^{\mathrm{d}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}} Z^{\mathrm{d}}_{\nu^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\right] \\
&= \mathbb{E}^{\mathrm{d}}\left[\mathbb{E}^{\mathrm{d}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\nu^{\mathrm{d}}}\right] Z^{\mathrm{d}}_{\nu^{\mathrm{d}}}\right] = \mathbb{E}\left[\int_0^\infty \mathbb{E}^{\mathrm{d}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\nu^{\mathrm{d}}}\right](u) Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(u)}(u)\eta(du)\right] \\
&= \mathbb{E}\left[\int_0^\infty \frac{\mathbb{E}^{\mathrm{d}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\nu^{\mathrm{d}}}\right](u)}{\alpha^{\mathrm{d}}_{\nu^{\mathrm{d}}(u)}(u)} Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_{\nu^{\mathrm{d}}(u)}(u)\eta(du)\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}\left[Z^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\nu^{\mathrm{d}}}\right](\tau)}{\alpha^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)} Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)\right]. \qquad (4.12)
\end{aligned}$$

Due to (4.12), we infer from Proposition 4.2.8 and the monotone class theorem that the first equality in (4.10) holds, i.e. $\mathbb{E}\left[\xi|\mathcal{G}_\nu\right] = \frac{\mathbb{E}^{\mathrm{d}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\nu^{\mathrm{d}}}\right](\tau)}{\alpha^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)}$. By Remark 4.2.5 i), we have $\widehat{\mathbb{P}} \sim \mathbb{P}\otimes\eta$ on $\widehat{\mathcal{F}}_{\nu^{\mathrm{d}}}$. An application of Bayes' formula shows that (4.10) holds.
We now prove the equality (4.11). Let Z^{d} a bounded $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable random variable. Proceeding as in (4.12), one obtains

$$\begin{aligned}
\mathbb{E}\left[\xi Z^{\mathrm{d}}_\nu(\tau)\right] &= \mathbb{E}\left[\int_0^\infty \xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u) Z^{\mathrm{d}}_\nu(u)\eta(du)\right] \\
&= \mathbb{E}\left[\int_0^\infty \frac{\mathbb{E}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}(u)|\mathcal{F}_\nu\right]}{\alpha^{\mathrm{d}}_\nu(u)} Z^{\mathrm{d}}_\nu(u)\alpha^{\mathrm{d}}_\nu(u)\eta(du)\right] \\
&= \mathbb{E}\left[\mathbb{E}\left[\frac{\mathbb{E}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}(u)|\mathcal{F}_\nu\right]|_{u=\tau}}{\alpha^{\mathrm{d}}_\nu(\tau)} Z^{\mathrm{d}}_\nu(\tau)\Big|\mathcal{F}_\nu\right]\right] \\
&= \mathbb{E}\left[\frac{\mathbb{E}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}(u)}}(u)\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}(u)|\mathcal{F}_\nu\right]|_{u=\tau}}{\alpha^{\mathrm{d}}_\nu(\tau)} Z^{\mathrm{d}}_\nu(\tau)\right].
\end{aligned}$$

From here, we conclude using similar arguments as for (4.10) that (4.11) holds. ☐

We recall that the survival process G is defined by $G_t = \mathbb{P}\left(\tau > t|\mathcal{F}_t\right),\ t \geq 0$. The following proposition is the counterpart of Proposition 4.2.17 for σ-algebras of the form $\mathcal{G}_\nu, \nu \in \mathcal{T}_T(\mathbb{G})$.

Proposition 4.2.18. *Let $\nu, \gamma \in \mathcal{T}_T(\mathbb{G})$ with $\nu \leq \gamma$, $\nu = \nu^b 1_{\{\nu^b<\tau\}} + \nu^d(\tau)1_{\{\nu^b\geq\tau\}}$ and $\gamma = \gamma^b 1_{\{\gamma^b<\tau\}} + \gamma^d(\tau)1_{\{\gamma^b\geq\tau\}}$. Let ξ^b (resp. ξ^d) be an $\mathcal{O}(\mathbb{F})$ (resp. $\mathcal{O}(\mathbb{F})\times\mathcal{B}(\mathbb{R}^+)$)-measurable random variable. Suppose that $\xi = \xi^b_{\gamma^b}1_{\{\gamma^b<\tau\}} + \xi^d_{\gamma^d(\tau)}(\tau)1_{\{\gamma^b\geq\tau\}} \in L^1(\Omega, \mathcal{G}_\gamma, \mathbb{P})$. Then*

$$\mathbb{E}\left[\xi|\mathcal{G}_\nu\right] = X^b 1_{\{\nu^b<\tau\}} + X^d(\tau)1_{\{\nu^b\geq\tau\}}, \qquad (4.13)$$

where

$$\begin{aligned}
X^b &= \frac{1}{G_{\nu^b}}\mathbb{E}\left[\xi^b_{\gamma^b}G_{\gamma^b} + \int_{\nu^b}^{\gamma^b\vee\nu^b} \xi^d_{\gamma^d(u)}(u)\alpha^d_{\gamma^d(u)}(u)\eta(du)\Big|\mathcal{F}_{\nu^b}\right] \qquad (4.14)\\
&= \frac{1}{G_{\nu^b}}\mathbb{E}\left[\xi^b_{\gamma^b}G_{\gamma^b} + \int_{\nu^b}^{\gamma^b\vee\nu^b} \widehat{\mathbb{E}}\left[\xi^d_{\gamma^d}|\widehat{\mathcal{F}}_{\tau^d_T}\right](u)\alpha^d_u(u)\eta(du)\Big|\mathcal{F}_{\nu^b}\right], \qquad (4.15)\\
X^d(\tau) &= \mathbb{E}\left[\xi^d_{\gamma^d(\tau)}(\tau)1_{\{\gamma^b\geq\tau\}}|\mathcal{G}^\tau_{\nu^d_T(\tau)}\right]. \qquad (4.16)
\end{aligned}$$

Proof. By Corollary 4.2.15, there exists a $\mathcal{G}_{\nu^{\mathrm{b}}}$-measurable r.v. X^{b} and a $\mathcal{G}^{\tau}_{\nu^{\mathrm{d}}_T(\tau)}$-measurable r.v. $X^{\mathrm{d}}(\tau)$ such that $\mathbb{E}\left[\xi|\mathcal{G}_{\nu}\right] = X^{\mathrm{b}}1_{\{\nu^{\mathrm{b}}<\tau\}} + X^{\mathrm{d}}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}$. We show that X^{b} and $X^{\mathrm{d}}(\tau)$ are given by (4.14) and (4.16) respectively.
We begin with (4.14). Multiplying (4.13) by $1_{\{\nu^{\mathrm{b}}<\tau\}}$ and taking conditional expectation w.r.t. $\mathcal{F}_{\nu^{\mathrm{b}}}$, we obtain $X^{\mathrm{b}}G_{\nu^{\mathrm{b}}} = \mathbb{E}\left[\mathbb{E}\left[\xi|\mathcal{G}_{\nu}\right]1_{\{\nu^{\mathrm{b}}<\tau\}}|\mathcal{F}_{\nu^{\mathrm{b}}}\right] = \mathbb{E}\left[\xi 1_{\{\nu^{\mathrm{b}}<\tau\}}|\mathcal{F}_{\nu^{\mathrm{b}}}\right]$. Indeed for $A\in\mathcal{F}_{\nu^{\mathrm{b}}}$, the random variable $1_A 1_{\{\nu^{\mathrm{b}}<\tau\}}$ is $\mathcal{G}_{\nu}$-measurable (Corollary 4.2.15) and

$$\mathbb{E}\left[\mathbb{E}\left[\xi|\mathcal{G}_{\nu}\right]1_{\{\nu^{\mathrm{b}}<\tau\}}1_A\right] = \mathbb{E}\left[\mathbb{E}\left[\xi 1_{\{\nu^{\mathrm{b}}<\tau\}}1_A\right]\right] = \mathbb{E}\left[\xi 1_{\{\nu^{\mathrm{b}}<\tau\}}1_A\right] = \mathbb{E}\left[\mathbb{E}\left[\xi 1_{\{\nu^{\mathrm{b}}<\tau\}}|\mathcal{F}_{\nu^{\mathrm{b}}}\right]1_A\right].$$

As $\gamma\geq\nu$, $\gamma^{\mathrm{b}} = \gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}$ on the event $\{\nu^{\mathrm{b}}<\tau\}$. This leads to the following structure of $\xi 1_{\{\nu^{\mathrm{b}}<\tau\}}$

$$\xi 1_{\{\nu^{\mathrm{b}}<\tau\}} = \xi^{\mathrm{b}}_{\gamma^{\mathrm{b}}}1_{\{\gamma^{\mathrm{b}}<\tau\}} + \xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)1_{\{\gamma^{\mathrm{b}}\geq\tau\}}1_{\{\nu^{\mathrm{b}}<\tau\}} = \xi^{\mathrm{b}}1_{\{\gamma^{\mathrm{b}}<\tau\}} + \xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)1_{\{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\geq\tau\}}1_{\{\nu^{\mathrm{b}}<\tau\}}.$$

Using the above equality and the equality $X^{\mathrm{b}}G_{\nu^{\mathrm{b}}} = \mathbb{E}\left[\xi 1_{\{\nu^{\mathrm{b}}<\tau\}}|\mathcal{F}_{\nu^{\mathrm{b}}}\right]$, we see that the density hypothesis yields

$$\begin{aligned}
X^{\mathrm{b}}G_{\nu^{\mathrm{b}}} &= \mathbb{E}\left[\xi^{\mathrm{b}}_{\gamma^{\mathrm{b}}}1_{\{\gamma^{\mathrm{b}}<\tau\}} + \xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)1_{\{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\geq\tau\}}1_{\{\nu^{\mathrm{b}}<\tau\}}|\mathcal{F}_{\nu^{\mathrm{b}}}\right]\\
&= \mathbb{E}\left[\xi^{\mathrm{b}}_{\gamma^{\mathrm{b}}}G_{\gamma^{\mathrm{b}}} + \int_0^{\infty}\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)1_{\{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\geq u\}}1_{\{u>\nu^{\mathrm{b}}\}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\eta(du)\middle|\mathcal{F}_{\nu^{\mathrm{b}}}\right]\\
&= \mathbb{E}\left[\xi^{\mathrm{b}}_{\gamma^{\mathrm{b}}}G_{\gamma^{\mathrm{b}}} + \int_{\nu^{\mathrm{b}}}^{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}}\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\eta(du)\right].
\end{aligned}$$

We see that (4.14) is verified. To obtain (4.15) it suffices to show that

$$\mathbb{E}\left[\int_{\nu^{\mathrm{b}}}^{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}}\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\eta(du)\middle|\mathcal{F}_{\nu^{\mathrm{b}}}\right] = \mathbb{E}\left[\int_{\nu^{\mathrm{b}}}^{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}}\widehat{\mathbb{E}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\tau^{\mathrm{d}}_T}\right](u)\alpha^{\mathrm{d}}_u(u)\eta(u)\middle|\mathcal{F}_{\nu^{\mathrm{b}}}\right]. \quad (4.17)$$

Let $A\in\mathcal{F}_{\nu^{\mathrm{b}}}$. We recall that $\gamma^{\mathrm{b}},\nu^{\mathrm{b}}\in\mathcal{T}_T(\mathbb{F})$ and $\mathbb{P}\otimes\eta\sim\widehat{\mathbb{P}}$ on $\widehat{\mathcal{F}}_{\tau^{\mathrm{d}}_T}$. Thus

$$\begin{aligned}
\mathbb{E}\left[\int_{\nu^{\mathrm{b}}}^{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}}\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}(u)}(u)\eta(du)1_A\right] &= \mathbb{E}^{\mathrm{d}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}1_{\{\nu^{\mathrm{b}}<\tau^{\mathrm{d}}\leq\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\}}1_A\right]\\
&= \mathbb{E}^{\mathrm{d}}\left[\mathbb{E}^{\mathrm{d}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}\alpha^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\tau^{\mathrm{d}}_T}\right]1_{\{\nu^{\mathrm{b}}<\tau^{\mathrm{d}}_T\leq\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\}}1_A\right]\\
&= \mathbb{E}^{\mathrm{d}}\left[\widehat{\mathbb{E}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\tau^{\mathrm{d}}_T}\right]\alpha^{\mathrm{d}}_{\tau^{\mathrm{d}}_T}1_{\{\nu^{\mathrm{b}}<\tau^{\mathrm{d}}_T\leq\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\}}1_A\right]\\
&= \mathbb{E}\left[\int_0^{\infty}\widehat{\mathbb{E}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\tau^{\mathrm{d}}_T}\right](u)\alpha^{\mathrm{d}}_u(u)1_{\{\nu^{\mathrm{b}}<u\leq\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}\}}1_A\eta(du)\right].
\end{aligned}$$

By the law of iterated conditional expectation, the right hand side of the above equality gives

$$\mathbb{E}\left[\mathbb{E}\left[\int_{\nu^{\mathrm{b}}}^{\gamma^{\mathrm{b}}\vee\nu^{\mathrm{b}}}\widehat{\mathbb{E}}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}}|\widehat{\mathcal{F}}_{\tau^{\mathrm{d}}_T}\right](u)\alpha^{\mathrm{d}}_u(u)\eta(du)\middle|\mathcal{F}_{\nu^{\mathrm{b}}}\right]1_A\right].$$

As A is arbitrary, we deduce that (4.17) holds. Hence (4.15) holds as well.
We proceed to show (4.16), or equivalently, $\mathbb{E}\left[\xi 1_{\{\nu^{\mathrm{b}}\geq\tau\}}|\mathcal{G}_{\nu}\right] = X^{\mathrm{d}}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}$. To this end, let $Z = Z_{\nu^{\mathrm{b}}}1_{\{\nu^{\mathrm{b}}<\tau\}} + Z^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}$ be a $\mathcal{G}_{\nu}$-measurable and bounded r.v.. Since $\nu\leq\gamma\leq T$, we have

$$\begin{aligned}
\mathbb{E}\left[\xi Z 1_{\{\nu^{\mathrm{b}}\geq\tau\}}\right] &= \mathbb{E}\left[\xi^{\mathrm{b}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)Z^{\mathrm{d}}_{\nu^{\mathrm{d}}_T(\tau)}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}\right] = \mathbb{E}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)1_{\{\gamma^{\mathrm{b}}\geq\tau\}}Z^{\mathrm{d}}_{\nu^{\mathrm{d}}_T(\tau)}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}\right]\\
&= \mathbb{E}\left[\mathbb{E}\left[\xi^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)}(\tau)1_{\{\gamma^{\mathrm{b}}\geq\tau\}}|\mathcal{G}^{\tau}_{\nu^{\mathrm{d}}_T(\tau)}\right]Z^{\mathrm{d}}_{\nu^{\mathrm{d}}_T(\tau)}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}\right] = \mathbb{E}\left[X^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}Z\right].
\end{aligned}$$

We infer from the monotone class theorem that $\mathbb{E}\left[\xi 1_{\{\nu^{\mathrm{b}}\geq\tau\}}|\mathcal{G}_{\nu}\right] = X^{\mathrm{d}}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}$ which yields (4.16). □

Remark 4.2.19. *Proposition 4.2.18 remains true if* $\xi^d_{\gamma^d(\tau)}(\tau) \in L^1(\Omega, \mathcal{G}^\tau_{\gamma^d(\tau)}, \mathbb{P})$ *and we take* $X^d(\tau) = \mathbb{E}\left[\xi^d_{\gamma^d(\tau)}(\tau) | \mathcal{G}^\tau_{\nu^d_T(\tau)}\right]$.

4.2.4 Characterization of supermartingales

Our goal in this section is to give a description of $(\mathbb{G}, \mathbb{P})$-supermartingales in terms of $\mathbb{F}$ and $\mathbb{G}^\tau$-supermartingales. For $(\mathbb{G}, \mathbb{P})$-local martingales, this characterization has been obtained in [EKJJ10, CJZ13]. The optimality principle for stochastic control problems is underpinned by the supermartingale property of the associated dynamic value process resulting from the dynamic programming principle, see [EK81]. We will rely on the description of $\mathbb{G}$-supermartingales in terms of $\mathbb{F}$-supermartingales and $\mathbb{G}^\tau$-supermartingales to identify the candidate dynamic value processes for the sub-control problems in $\mathbb{F}$ and $\mathbb{G}^\tau$ faced by agents respectively before and after τ.

We recall some classical definitions that will be used throughout. As we deal with different filtrations and measures, we give the definitions for some generic filtered probability space $\Big(\Sigma, \mathbb{A}, \mathbb{H} = (\mathcal{H}_t)_{t\in[0,T]}, \mathbb{Q}\Big)$.

Definition 4.2.20. *Let* $Z = (Z_t)_{t\in[0,T]}$ *be an* $\mathbb{H}$*-optional process. We say that* Z *is of class* (D) *w.r.t. the measure* $\mathbb{Q}$ *if the family of random variables* $\{Z_\delta, \delta \in \mathcal{T}_T(\mathbb{H})\}$ *is* $\mathbb{Q}$*-uniformly integrable. We denote by* $\mathcal{D}_T(\mathbb{H}, \mathbb{Q})$ *the set of* $\mathbb{H}$*-adapted optional processes of class* (D) *w.r.t. the measure* $\mathbb{Q}$.

Definition 4.2.21. *A real valued càdlàg* $\mathbb{H}$*-adapted process* $(Z_t)_{t\in[0,T]}$ *is a* special semimartingale *if and only if it admits a decomposition of the form*

$$Z_t \;=\; M_t + A_t,\;\; t \in [0,T],$$

where M *is an* $(\mathbb{H}, \mathbb{Q})$*-local martingale and* A *is an* $\mathbb{H}$*-predictable process with paths of finite variation.*

The following proposition links $\mathbb{G}^\tau$-supermartingales w.r.t. $\widehat{\mathbb{F}}$ and $\mathbb{F}$- supermartingales. A similar result holds for submartingales. We recall that $\widehat{\Omega} = \Omega \times \mathbb{R}^+$.

Proposition 4.2.22. *Let* $X^d : [0,T] \times \widehat{\Omega} \to \mathbb{R}$ *be an* $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$*-measurable process. The following assertions are equivalent:*

i) $\Big(X^d_t(\tau)\Big)_{t\in[0,T]}$ *is a càdlàg* $(\mathbb{G}^\tau, \mathbb{P})$*-supermartingale.*

ii) $(X^d)_{t\in[0,T]}$ *is a càdlàg* $(\widehat{\mathbb{F}}, \widehat{\mathbb{P}})$*-supermartingale.*

iii) $(X^d_t(u)\alpha^d_t(u))_{t\in[0,T]}$ *is a càdlàg* $(\mathbb{F}, \mathbb{P})$*-supermartingale for* η*-almost all* $u \in [0,T]$.

Proof. Let $\gamma^{\mathrm{d}}, \nu^{\mathrm{d}} \in \mathcal{T}_T(\widehat{\mathbb{F}})$ with $\nu^{\mathrm{d}} \le \gamma^{\mathrm{d}}$ $\widehat{\mathbb{P}}$-a.s.. By Proposition 4.2.17, $\mathbb{E}\left[X^{\mathrm{d}}_{\gamma^{\mathrm{d}}(\tau)} | \mathcal{G}^\tau_{\nu^{\mathrm{d}}(\tau)}\right] \le X^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)$ if and only if $\widehat{\mathbb{E}}\left[X^{\mathrm{d}}_{\gamma^{\mathrm{d}}} | \widehat{\mathcal{F}}_{\nu^{\mathrm{d}}}\right](\tau) \le X^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)$. The equivalence between i) and ii) is a consequence of Remark 4.2.5 ii). By Remark 4.2.5 i), $\widehat{\mathbb{P}}$ is equivalent to $\mathbb{P}\otimes\eta$ on $\widehat{\mathcal{F}}_T$ with Radon-Nikodym density α^{d}_T. Using Bayes' formula, one can see that i) is equivalent to: $(X^{\mathrm{d}}_t \alpha^{\mathrm{d}}_t)_{t\in[0,T]}$ is an $(\widehat{\mathbb{F}}, \mathbb{P}\otimes\eta)$-supermartingale. By Fubini's theorem, i) and iii) are equivalent. □

Next we give a characterization of $(\mathbb{G}, \mathbb{P})$-supermartingales with càdlàg paths in terms of $(\mathbb{F}, \mathbb{P})$-supermartingales. Analogous assertions are valid as well for $(\mathbb{G}, \mathbb{P})$-submartingales and $(\mathbb{G}, \mathbb{P})$-martingales with càdlàg paths.

Proposition 4.2.23. *Let $r>1$ and $Y \in \mathcal{S}^r_T(\mathbb{G},\mathbb{P})$ with $Y_t = Y^b_t 1_{\{t<\tau\}} + Y^d_t(\tau)1_{\{t\geq\tau\}}, t\in[0,T]$, where Y^b is an $\mathbb{F}$-adapted càdlàg process and Y^d is $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable. Let $\nu\in\mathcal{T}_T(\mathbb{G})$. Let $\nu = \nu^b 1_{\{\nu^b<\tau\}} + \nu^d(\tau)1_{\{\nu^b\geq\tau\}} \in \mathcal{T}_T(\mathbb{G})$, where $\nu^b\in\mathcal{T}(\mathbb{F})$ and $\nu^d\in\mathcal{T}_T(\widehat{\mathbb{F}})$. We consider the process $A = Y^bG + \int_0^\cdot Y^d_u(u)\alpha^d_u(u)\eta(du)$ with G the survival process defined by (4.2). The following hold:*

a) The stopped process $Y_{\cdot\wedge\tau}$ is a $(\mathbb{G},\mathbb{P})$-supermartingale if and only if the process A is an $(\mathbb{F},\mathbb{P})$-supermartingale.

b) Y is a $(\mathbb{G},\mathbb{P})$-supermartingale if and only if

i) the stopped process $Y_{\cdot\wedge\tau}$ is a $(\mathbb{G},\mathbb{P})$-supermartingale,

ii) $\left(Y^d_t(u)\alpha^d_t(u)\right)_{t\geq u}$ is an $(\mathbb{F},\mathbb{P})$-supermartingale for η-almost all $u\in[0,T]$.

c) If Y is a $(\mathbb{G},\mathbb{P})$-supermartingale and $Y_{\cdot\wedge\nu}$ a $(\mathbb{G},\mathbb{P})$-martingale, then

i) The stopped process $A_{\cdot\wedge\nu^b}$ is an $(\mathbb{F},\mathbb{P})$-martingale,

ii) $\mathbb{E}\left[Y^d_{\nu^d(\tau)}(\tau)1_{\{\nu^b\geq\tau\}}\middle|\mathcal{G}^{\tau}_{T\wedge\tau}\right] = Y^d_\tau(\tau)1_{\{\nu^b\geq\tau\}}$.

For the proof, we need the following lemma which justifies the integrability of the process A.

Lemma 4.2.24. *We keep the notation and hypotheses of Proposition 4.2.23. Let $p=\frac{2r}{r+1}$. Then:*

a) $\int_0^\cdot |Y^d_u(u)|\alpha^d_u(u)\eta(du)\in\mathcal{S}^p_T(\mathbb{F},\mathbb{P})$,

b) $Y^bG\in\mathcal{S}^p_T(\mathbb{F},\mathbb{P})$,

c) $A = Y^bG + \int_0^\cdot Y^d_u(u)\alpha^d_u(u)\eta(du)\in\mathcal{S}^p_T(\mathbb{F},\mathbb{P})$.

Proof. a) Since $Y\in\mathcal{S}^r_T(\mathbb{G},\mathbb{P})$, we have $\mathbb{E}\left[|Y^{\mathrm{d}}_\tau(\tau)|^r 1_{\{\tau\leq T\}}\right]<+\infty$ and density hypothesis implies that

$$\mathbb{E}\left[\int_0^T |Y^{\mathrm{d}}_u(u)|^r\alpha^{\mathrm{d}}_u(u)\eta(du)\right]<+\infty. \tag{4.18}$$

Let $l_1=\frac{r+1}{2}$ and $l_2=\frac{r+1}{r-1}$. Applying Hölder's inequality, one obtains

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}\left|\int_0^t Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)\right|^p\right] &\leq \mathbb{E}\left[\left(\int_0^T |Y^{\mathrm{d}}_u(u)||\alpha^{\mathrm{d}}_u(u)|^{\frac{1}{r}}|\alpha^{\mathrm{d}}_u(u)|^{\frac{r-1}{r}}\eta(du)\right)^p\right]\\
&\leq \mathbb{E}\left[\left(\int_0^T |Y^{\mathrm{d}}_u(u)|^r\alpha^{\mathrm{d}}_u(u)\eta(du)\right)^{\frac{2}{r+1}}\left(\int_0^T \alpha^{\mathrm{d}}_u(u)\eta(du)\right)^{\frac{2(r-1)}{r+1}}\right]\\
&\leq \left(\mathbb{E}\left[\int_0^T |Y^{\mathrm{d}}_u(u)|^r\alpha^{\mathrm{d}}_u(u)\eta(du)\right]\right)^{l_1}\left(\mathbb{E}\left[\left(\int_0^T \alpha^{\mathrm{d}}_u(u)\eta(du)\right)^2\right]\right)^{l_2}.
\end{aligned}$$

As $\mathbb{E}\left[\left(\int_0^T\alpha^{\mathrm{d}}_u(u)\eta(du)\right)^2\right]<+\infty$ (see [EKJJ10]), we deduce from the above inequalities that a) holds.
We now show b). Let $\kappa=\frac{r}{p}\in(1,+\infty)$. By the fact that $G\leq 1$, it follows that for $t\in[0,T]$:

$$|Y^{\mathrm{b}}_tG_t|^p \;\leq\; |Y^{\mathrm{b}}_t|^pG_t = \mathbb{E}\left[|Y^{\mathrm{b}}_t|^p 1_{\{t<\tau\}}\middle|\mathcal{F}_t\right] = \mathbb{E}\left[|Y_t|^p 1_{\{t<\tau\}}\middle|\mathcal{F}_t\right] \leq \mathbb{E}\left[\sup_{s\in[0,T]}|Y_s|^p\middle|\mathcal{F}_t\right].$$

Let M be the martingale defined by $M_t = \mathbb{E}\left[\sup_{s\in[0,T]} |Y_s|^p|\mathcal{F}_t\right]$, $t \in [0,T]$. Clearly $M \in \mathcal{S}^\kappa_T(\mathbb{F},\mathbb{P})$ since $\mathbb{E}\left[|\sup_{t\in[0,T]} |Y_t|^r\right] < +\infty$. Applying Doob's maximal inequality, we obtain

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y^{\mathrm{b}}_t G_t|^p\right] \leq \mathbb{E}\left[\sup_{t\in[0,T]} |M_t|\right] \leq \left(\mathbb{E}\left[\sup_{t\in[0,T]} |M^\kappa_t|\right]\right)^{\frac{1}{\kappa}} < \frac{\kappa}{\kappa-1}\mathbb{E}\left[M^\kappa_T\right] < +\infty.$$

Thus $Y^{\mathrm{b}}G \in \mathcal{S}^p_T(\mathbb{F},\mathbb{P})$. c) is a consequence of a) and b). □

Proof of Proposition 4.2.23. Note that $A \in \mathcal{D}_T(\mathbb{F},\mathbb{P})$ due to Lemma 4.2.24.
a) $'' \Rightarrow ''$Let $s,t \in [0,T], s \leq t$. Then $Y_{t\wedge\tau} = Y^{\mathrm{b}}_t 1_{\{t<\tau\}} + Y^{\mathrm{d}}_\tau(\tau)1_{\{t\geq\tau\}}$. Proposition 4.2.18 yields

$$\mathbb{E}\left[Y_{t\wedge\tau}|\mathcal{G}_{s\wedge\tau}\right] = \frac{1}{G_s}\mathbb{E}\left[Y^{\mathrm{b}}_t G_t + \int_s^t Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)\middle|\mathcal{F}_s\right]1_{\{s<\tau\}} + Y^{\mathrm{d}}_\tau(\tau)1_{\{s\geq\tau\}}.$$

The supermartingale inequality[3] $\mathbb{E}\left[Y_{t\wedge\tau}|\mathcal{G}_{s\wedge\tau}\right]1_{\{s<\tau\}} \leq Y^{\mathrm{b}}_s 1_{\{s<\tau\}}$ and the above equality imply that

$$\frac{1}{G_s}\mathbb{E}\left[Y^{\mathrm{b}}_t G_t + \int_s^t Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)\middle|\mathcal{F}_s\right]1_{\{s<\tau\}} \leq Y^{\mathrm{b}}_s 1_{\{s<\tau\}}.$$

Taking the conditional expectation w.r.t. $\mathcal{F}_s$ we obtain

$$\mathbb{E}\left[Y^{\mathrm{b}}_t G_t + \int_s^t Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)\middle|\mathcal{F}_s\right] \leq Y^{\mathrm{b}}_s G_s.$$

One infers that $\mathbb{E}\left[A_t|\mathcal{F}_s\right] \leq A_s$. As s,t are arbitrary, A is an $(\mathbb{F},\mathbb{P})$-supermartingale.
$'' \Leftarrow ''$ Let $s,t \in [0,T], s \leq t$. We have $\mathbb{E}\left[A_t|\mathcal{F}_s\right] \leq A_s$ and therefore

$$\begin{aligned}
\mathbb{E}\left[Y_{t\wedge\tau}|\mathcal{G}_s\right] &= \mathbb{E}\left[Y_{t\wedge\tau}|\mathcal{G}_{s\wedge\tau}\right]1_{\{s<\tau\}} + \mathbb{E}\left[Y_{t\wedge\tau}|\mathcal{G}_s\right]1_{\{s\geq\tau\}} \\
&= \frac{1}{G_s}\mathbb{E}\left[Y^{\mathrm{b}}_t G_t + \int_s^t Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)\middle|\mathcal{F}_s\right]1_{\{s<\tau\}} + Y^{\mathrm{d}}_\tau(\tau)1_{\{s\geq\tau\}} \\
&\leq Y^{\mathrm{b}}_s 1_{\{s<\tau\}} + Y^{\mathrm{d}}_\tau(\tau)1_{\{\tau\geq s\}}.
\end{aligned}$$

We deduce that $\mathbb{E}\left[Y_{t\wedge\tau}|\mathcal{G}_s\right] \leq Y_{s\wedge\tau}$. As s,t are arbitrary , $Y_{\cdot\wedge\tau}$ is a $(\mathbb{G},\mathbb{P})$-supermartingale.
b)$'' \Rightarrow ''$. The process $Y_{\cdot\wedge\tau}$ is a $(\mathbb{G},\mathbb{P})$-supermartingale since Y is a $(\mathbb{G},\mathbb{P})$-supermartingale. To prove ii) we apply Proposition 4.2.18. For $s,t \in [0,T]$ with $s \leq t$, we have

$$\mathbb{E}\left[Y_t|\mathcal{G}_s\right]1_{\{s\geq\tau\}} = \frac{\mathbb{E}\left[Y^{\mathrm{d}}_t(u)\alpha^{\mathrm{d}}_t(u)|\mathcal{F}_s\right]|_{u=\tau}}{\alpha^{\mathrm{d}}_s(\tau)}1_{\{s\geq\tau\}} \leq Y^{\mathrm{d}}_s(\tau)1_{\{s\geq\tau\}}.$$

The above inequality entails that for η-almost all $u \leq s \leq t, \mathbb{E}\left[Y^{\mathrm{d}}_t(u)\alpha^{\mathrm{d}}_t(u)|\mathcal{F}_s\right] \leq Y^{\mathrm{d}}_s(u)\alpha^{\mathrm{d}}_s(u)$.
Hence $\left(Y^{\mathrm{d}}_t(u)\alpha^{\mathrm{d}}_t(u)\right)_{t\geq u}$ is an $(\mathbb{F},\mathbb{P})$-supermartingale for η-almost all $u \in \mathbb{R}^+$.
$'' \Leftarrow ''$ By Propositions 4.2.18 and 4.2.17, for $s,t \in [0,T]$ with $s \leq t$ we have

$$\mathbb{E}\left[Y_t|\mathcal{G}_s\right]1_{\{s\geq\tau\}} = \frac{\mathbb{E}\left[Y^{\mathrm{d}}_t(u)\alpha^{\mathrm{d}}_t(u)|\mathcal{F}_s\right]|_{u=\tau}}{\alpha^{\mathrm{d}}_s(\tau)}1_{\{s\geq\tau\}} \leq Y^{\mathrm{d}}_s 1_{\{s\geq\tau\}}.$$

Using Proposition 4.2.18, a) and ii), one obtains the inequality

$$\mathbb{E}\left[Y_t|\mathcal{G}_s\right]1_{\{s<\tau\}} = \frac{1}{G_s}\mathbb{E}\left[Y^{\mathrm{b}}_t G_t + \int_s^t \mathbb{E}\left[Y^{\mathrm{d}}_t(u)\alpha^{\mathrm{d}}_t(u)|\mathcal{F}_u\right]\eta(du)\middle|\mathcal{F}_s\right]1_{\{s<\tau\}} \leq Y^{\mathrm{b}}_s 1_{\{s<\tau\}}.$$

[3] As Y is a $\mathbb{G}$-supermartingale, $Y_{\cdot\wedge\tau}$ is adapted to the filtration $(\mathcal{G}_{s\wedge\tau})_{s\in[0,T]}$ and it is also a supermartingale w.r.t. this filtration.

Combining the above two inequalities, we see that the supermartingale property for Y is proved.
c) Note that A is an $(\mathbb{F},\mathbb{P})$-supermartingale as Y is a $(\mathbb{G},\mathbb{P})$-supermartingale. $A_{\cdot\wedge\nu^{\mathrm{b}}}$ is therefore an $(\mathbb{F},\mathbb{P})$-supermartingale. Hence to show that $A_{\cdot\wedge\nu^{\mathrm{b}}}$ is an $(\mathbb{F},\mathbb{P})$-martingale, it will be sufficient to show the equality $\mathbb{E}\left[A_{\nu^{\mathrm{b}}}\right]=\mathbb{E}\left[A_0\right]=\mathbb{E}\left[Y_0^{\mathrm{b}}\right]$. To this end, we employ the martingale property of $Y_{\cdot\wedge\nu}$ which entails that $\mathbb{E}\left[Y_{\nu\wedge\tau}\right]=\mathbb{E}\left[Y_0\right]=\mathbb{E}\left[Y_0^{\mathrm{b}}\right]$. Clearly $Y_{\nu\wedge\tau}=Y^{\mathrm{b}}_{\nu^{\mathrm{b}}}1_{\{\nu^{\mathrm{b}}<\tau\}}+Y^{\mathrm{d}}_{\tau}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}$ and the density hypothesis gives $\mathbb{E}\left[Y_{\nu\wedge\tau}\right]=\mathbb{E}\left[Y^{\mathrm{b}}_{\nu^{\mathrm{b}}}G_{\nu^{\mathrm{b}}}+\int_0^{\nu^{\mathrm{b}}}Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)\right]=\mathbb{E}\left[A_{\nu^{\mathrm{b}}}\right]$. Thus i) holds.
To obtain ii), we rely once more on the martingale property of $Y_{\cdot\wedge\nu}$ which yield the equality $\mathbb{E}\left[Y_\nu|\mathcal{G}_{\nu\wedge\tau}\right]=Y_{\nu\wedge\tau}$. By Proposition 4.2.18 the last equality can be written equivalently as

$$Y^{\mathrm{b}}_{\nu^{\mathrm{b}}}1_{\{\nu^{\mathrm{b}}<\tau\}}+\mathbb{E}\left[Y^{\mathrm{d}}_{\nu^{\mathrm{d}}(\tau)}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}|\mathcal{G}^{\tau}_{T\wedge\tau}\right]1_{\{T\geq\tau\}} \quad=\quad Y^{\mathrm{b}}_{\nu^{\mathrm{b}}}1_{\{\nu^{\mathrm{b}}<\tau\}}+Y^{\mathrm{d}}_{\tau}(\tau)1_{\{\nu^{\mathrm{b}}\geq\tau\}}.$$

From the inclusion $\{\nu^{\mathrm{b}}\geq\tau\}\subseteq\{T\geq\tau\}$, we deduce that ii) holds. □

Remark 4.2.25. *By Proposition 4.2.22, the assertion ii) in part b) of Proposition 4.2.23 is equivalent to the* $(\mathbb{G}^\tau,\mathbb{P})$*-supermartingale property of* $(Y_t^d(\tau)1_{\{t\geq\tau\}})_{t\in[0,T]}$.

The following proposition gives an equivalent characterization of the supermartingale property of the process A in Proposition 4.2.23 which involves the local martingale $L^{\mathbb{F}}$ (see Proposition 4.2.4) and the jump of Y at τ. It will be used in Section 4.3.4 where Y is the dynamic value process associated to the optimal stopping problem to obtain a recursive formula of Y^{b}.

Proposition 4.2.26. *We keep the notation and hypotheses of Proposition 4.2.23. Let* $p=\frac{2r}{r+1}$. *Let* $\lambda^{\mathbb{F}}$ *and* $L^{\mathbb{F}}$ *be as in Proposition 4.2.4. Recall that* $G=L^{\mathbb{F}}e^{-\int_0^{\cdot}\lambda^{\mathbb{F}}_s\eta(ds)}$. *Assume that* $r>1$ *and there exists* $l>\frac{p}{p-1}$ *such that* $\mathbb{E}\left[e^{l\int_0^T\lambda^{\mathbb{F}}_u\eta(du)}\right]<+\infty$. *The following assertions are equivalent:*

i) $A=Y^bG+\int_0^{\cdot}Y^d_u(u)\alpha^d_u(u)\eta(du)$ *is a càdlàg* $(\mathbb{F},\mathbb{P})$*-supermartingale (resp. submartingale).*

ii) $B=Y^bL^{\mathbb{F}}+\int_0^{\cdot}(Y^d_u(u)-Y^b_{u-})L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)$ *is a càdlàg* $(\mathbb{F},\mathbb{P})$*-supermartingale (resp. submartingale).*

Proof. First we will show that the process $B\in\mathcal{D}_T(\mathbb{F},\mathbb{P})$. To this end, let $\delta\in(1,\frac{pl}{p+l})$. We will show that $B\in\mathcal{S}^{\delta}_T(\mathbb{F},\mathbb{P})$. Let $\beta_1=\frac{p}{\delta}$ and $\beta_2=\frac{\beta_1}{\beta_1-1}=\frac{p}{p-\delta}$. Note that $\beta_2\delta<l$. We recall that for $t\in[0,T], G_t=L^{\mathbb{F}}e^{-\int_0^t\lambda^{\mathbb{F}}_u\eta(du)}$ and $\lambda^{\mathbb{F}}_t=\frac{\alpha^{\mathrm{d}}_t(t)}{G_t}$. Now For $t\in[0,T], |Y^{\mathrm{b}}_tL^{\mathbb{F}}_t|\leq|Y^{\mathrm{b}}_tG_t|e^{\int_0^T\lambda^{\mathbb{F}}_u\eta(du)}$. We infer from Hölder's inequality that

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|Y^{\mathrm{b}}_tL^{\mathbb{F}}_t|^{\delta}\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]}|Y^{\mathrm{b}}_tG_t|^{\delta}e^{\delta\int_0^T\lambda^{\mathbb{F}}_u\eta(du)}\right]\\
&\leq \left(\mathbb{E}\left[\sup_{t\in[0,T]}|Y^{\mathrm{b}}_tG_t|^{\beta_1\delta}\right]\right)^{\frac{1}{\beta_1}}\left(\mathbb{E}\left[e^{\beta_2\delta\int_0^T\lambda^{\mathbb{F}}_u\eta(du)}\right]\right)^{\frac{1}{\beta_2}}\\
&\leq \left(\mathbb{E}\left[\sup_{t\in[0,T]}|Y^{\mathrm{b}}_tG_t|^{p}\right]\right)^{\frac{1}{\beta_1}}\left(\mathbb{E}\left[e^{\beta_2\delta\int_0^T\lambda^{\mathbb{F}}_u\eta(du)}\right]\right)^{\frac{1}{\beta_2}}.
\end{aligned}$$

We have $\beta_2\delta<l$ and $Y^{\mathrm{b}}G\in\mathcal{S}^p_t(\mathbb{F},\mathbb{P})$ by Lemma 4.2.24. The right hand term in the last line of the inequality is therefore finite and $Y^{\mathrm{b}}L^{\mathbb{F}}\in\mathcal{S}^{\delta}_T(\mathbb{F},\mathbb{P})$. It remains to show that the process $\int_0^{\cdot}(Y^{\mathrm{d}}_u(u)-Y^{\mathrm{b}}_{u-})\lambda^{\mathbb{F}}_uL^{\mathbb{F}}_u\eta(du)\in\mathcal{S}^{\delta}_T(\mathbb{F},\mathbb{P})$. Using the relation $L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u=\alpha^{\mathrm{d}}_u(u)e^{\int_0^u\lambda^{\mathbb{F}}_s\eta(ds)}, u\in[0,T]$

and similar arguments as above give

$$\mathbb{E}\left[\left(\int_0^T |Y^{\mathrm{d}}_u(u)|L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)\right)^\delta\right] \le \mathbb{E}\left[e^{\int_0^T \delta\lambda^{\mathbb{F}}_u\eta(du)}\left(\int_0^T |Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)|\eta(du)\right)^\delta\right]$$
$$\le \left(\mathbb{E}\left[e^{\beta_2\delta\int_0^T \lambda^{\mathbb{F}}_u\eta(du)}\right]\right)^{\frac{1}{\beta_2}}\left(\mathbb{E}\left[\left(\int_0^T |Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)|\eta(du)\right)^p\right]\right)^{\frac{1}{\beta_1}}.$$

The last inequality is finite due to Lemma 4.2.24 and the fact that $\beta_2\delta < l$. Employing once more similar arguments as above we obtain

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^T |Y^{\mathrm{b}}_{u-}|\lambda^{\mathbb{F}}_u L^{\mathbb{F}}_u\eta(du)\right)^\delta\right] &= \mathbb{E}\left[\left(\int_0^T |Y^{\mathrm{b}}_{u-}|G_u\lambda^{\mathbb{F}}_u e^{\int_0^u \lambda^{\mathbb{F}}_s\eta(ds)}\eta(du)\right)^\delta\right]\\
&\le \mathbb{E}\left[\sup_{t\in[0,T]}|Y^{\mathrm{b}}_t G_t|^\delta\left(\int_0^T \lambda^{\mathbb{F}}_u e^{\int_0^u \lambda^{\mathbb{F}}_s\eta(ds)}\eta(du)\right)^\delta\right]\\
&\le \mathbb{E}\left[\sup_{t\in[0,T]}|Y^{\mathrm{b}}_t G_t|^\delta\left(e^{\int_0^T \lambda^{\mathbb{F}}_u\eta(du)}-1\right)^\delta\right] < +\infty.\end{aligned}$$

Combining the last two inequalities, we see that $\int_0^\cdot (Y^{\mathrm{d}}_u(u) - Y^{\mathrm{b}}_{u-})\lambda^{\mathbb{F}}_u L^{\mathbb{F}}_u\eta(du) \in \mathcal{S}^\delta_T(\mathbb{F},\mathbb{P})$.

We now ready to show the equivalences between i) and ii). The idea of the proof is to determine the dynamics of B in terms of A and rely on the Doob-Meyer's decomposition theorem of supermartingales to show that locally A and B are $(\mathbb{F},\mathbb{P})$-supermartingales. An application of Itô's formula yields for $t \in [0,T]$

$$\begin{aligned}d\left(Y^{\mathrm{b}}_t L^{\mathbb{F}}_t\right) &= d\left(Y^{\mathrm{b}}_t G_t e^{\int_0^t \lambda^{\mathbb{F}}_s\eta(ds)}\right) = d\left(A_t - \int_0^t Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)\right)e^{\int_0^t \lambda^{\mathbb{F}}_s\eta(ds)}\\
&=(A_{t-} - \int_0^t Y^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u\eta(du))\lambda^{\mathbb{F}}_t e^{\int_0^t \lambda^{\mathbb{F}}_u\eta(du)}\eta(dt) + e^{\int_0^t \lambda^{\mathbb{F}}_u\eta(du)}(dA_t - Y^{\mathrm{d}}_t(t)\alpha^{\mathrm{d}}_t(t)\eta(dt))\\
&= e^{\int_0^t \lambda^{\mathbb{F}}_s\eta(ds)}dA_t + \left((Y^{\mathrm{b}}_{t-}G_{t-}e^{\int_0^t \lambda^{\mathbb{F}}_u\eta(du)} - Y^{\mathrm{d}}_t(t)G_t e^{\int_0^t \lambda^{\mathbb{F}}_u\eta(du)}\right)\lambda^{\mathbb{F}}_t\eta(dt)\\
&= e^{\int_0^t \lambda^{\mathbb{F}}_s\eta(ds)}dA_t - \left(Y^{\mathrm{d}}_t(t) - Y^{\mathrm{b}}_{t-}\right)L^{\mathbb{F}}_t\lambda^{\mathbb{F}}_t\eta(dt).\end{aligned}$$

Hence $dB = e^{\int_0^\cdot \lambda^{\mathbb{F}}_s\eta(ds)}dA$. We prove the assertion for supermartingales.

i) $\Rightarrow$ ii). As A is an $(\mathbb{F},\mathbb{P})$-supermartingale with càdlàg paths, it admits a Doob-Meyer decomposition $A = M^A - K^A$ where M^A is an $(\mathbb{F},\mathbb{P})$-local martingale and K^A an integrable $\mathbb{F}$-predictable increasing process of finite variation starting at 0. Since η is non-atomic, $e^{\int_0^\cdot \lambda^{\mathbb{F}}\eta(ds)}$ is continuous and thus locally bounded. The semimartingale B is therefore special[4] and [CMS80, Théoreme 2] entails that $B = B_0 + M^B - K^B$ where $M^B_t = \int_0^t e^{\int_0^s \lambda^{\mathbb{F}}_u\eta(du)}dM^A_t$ and $K^B = \int_0^t e^{\int_0^s \lambda^{\mathbb{F}}_u\eta(du)}dK^A_s$ for $t \in [0,T]$. M^B being an $\mathbb{F}$-local martingale and K^B an $\mathbb{F}$-predictable process of finite variation, there exists an increasing sequence of stopping times $(\sigma_n)_{n\in\mathbb{N}} \subseteq \mathcal{T}_T(\mathbb{F})$ such that $\sigma_n \uparrow T$ as $n \to +\infty$, K_{σ_n} is $\mathbb{P}$-integrable and the stopped process $M^B_{\cdot\wedge\sigma_n}$ is an $(\mathbb{F},\mathbb{P})$-martingale. Clearly $\mathbb{E}\left[B_{t\wedge\sigma_n}|\mathcal{F}_s\right] \le B_{s\wedge\sigma_n}$ for $s,t \in [0,T]$ with $s \le t$. Since $B \in \mathcal{D}_T(\mathbb{F},\mathbb{P})$, taking the limit in the last inequality we obtain $\mathbb{E}\left[B_t|\mathcal{F}_s\right] \le B_s$ for $s,t \in [0,T]$ with $s \le t$. Hence B is a càdlàg $(\mathbb{F},\mathbb{P})$-supermartingale.

ii) $\Rightarrow$ i). Lemma 4.2.24 ensures that $A \in \mathcal{D}_T(\mathbb{F},\mathbb{P})$. By Ito's formula, $dA = e^{-\int_0^\cdot \lambda^{\mathbb{F}}_u\eta(du)}dB$. One proceeds using the same arguments as in the implication i) $\Rightarrow$ ii) to show that A is an $(\mathbb{F},\mathbb{P})$-supermartingale with càdlàg paths. $\square$

[4] Since $e^{\int_0^\cdot \lambda^{\mathbb{F}}_u\eta(du)}$ is locally bounded and A is special, the process $J_t = \sup_{s\le t}|\Delta_s B| \le e^{\int_0^t \lambda^{\mathbb{F}}_u\eta(du)}\sup_{s\le t}|\Delta_s A|$ is locally integrable.

Remark 4.2.27. *By Propositions 4.2.23 i) and 4.2.26, the stopped process $Y_{\cdot\wedge\tau}$ is a $(\mathbb{G},\mathbb{P})$-supermartingale if and only if B is an $(\mathbb{F},\mathbb{P})$-supermartingale. Proposition 4.2.26 generalizes the characterization of $(\mathbb{G},\mathbb{P})$-local martingales stopped at τ obtained in [EKJJ10, Proposition 5.1]. Note however that for local martingales, the integrability condition on $e^{\int_0^T \lambda_u^{\mathbb{F}}\eta(du)}$ is not required since there is no need to justify that B or A is integrable.*

4.3 Decomposition of stopping problems in the filtration $\mathbb{G}$

In this section, we consider optimal stopping problem faced by an agent with information flow given by $\mathbb{G}$. Due to the split of the information flow at time τ, the actions of agents before and after τ differ. Our goal is to identify stopping problems in the filtrations $\mathbb{F}$ and $\mathbb{G}^\tau$ corresponding respectively to the stopping problems faced by the agents before and after τ. This identification leads us to solve these problems in two steps, yielding a precise description of the actions of the agents in the presence of default. Optimal stopping problems appear in finance when dealing with the pricing and hedging of American contingent claims (see [Ben84, Kar88]). We will illustrate the importance of this two step methodology for addressing stopping problems by providing explicit formulae for hedges against such claims in Section 4.4. A further application of this methodology will be given in the following chapter where we address the solvability of reflected BSDEs in the filtration $\mathbb{G}$.

4.3.1 Preliminaries on the optimal stopping problem

We recall the formulation and the main results of the optimal stopping problem, see [EK81, KQ12]. As we will deal with different filtrations and measures, we consider here a probability measure $\mathbb{Q}$ on $(\Omega,\mathcal{A})$ which is equivalent to $\mathbb{P}$ and a filtration $\mathbb{H}=(\mathcal{H}_t)_{t\in[0,T]}$ on $(\Omega,\mathcal{A})$ satisfying the usual conditions. We recall the following useful definitions of semi-continuity.

Definition 4.3.1. *An $\mathbb{H}$-optional process ζ is said to be left-upper semicontinuous (resp. right-upper semicontinuous) in expectation along stopping times if for every $\gamma\in\mathcal{T}_T(\mathbb{H})$ and every sequence of $\mathbb{H}$-stopping times $(\gamma^n)_{n\in\mathbb{N}}$ such that $\gamma^n\uparrow\gamma$ (resp. $\gamma^n\downarrow\gamma$), we have*

$$\mathbb{E}^{\mathbb{Q}}[\zeta_\gamma] \quad\geq\quad \limsup_{n\to\infty}\mathbb{E}^{\mathbb{Q}}[\zeta_{\gamma_n}].$$

An $\mathbb{H}$-optional process $\zeta\in\mathcal{D}_T(\mathbb{H},\mathbb{Q})$ is called regular if it is left-continuous in expectation, i.e. for every $\gamma\in\mathcal{T}_T(\mathbb{H})$ and every sequence $(\gamma^n)_{n\in\mathbb{N}}\subseteq\mathcal{T}_T(\mathbb{H})$ such that $\gamma^n\uparrow\gamma$, we have

$$\mathbb{E}^{\mathbb{Q}}[\zeta_\gamma] \quad=\quad \lim_{n\to+\infty}\mathbb{E}^{\mathbb{Q}}[\zeta_{\gamma^n}].$$

The above definition of regularity is equivalent to ${}^pX=X_-$ where X_- is the left limit of X and ${}^pX$ the $\mathbb{H}$-predictable projection of X (see comments [DM82a, VI.50]).

4.3.2 The optimal stopping problem and the Snell envelope

Let us now formulate the optimal stopping problem. Let $\zeta=(\zeta_t)_{t\in[0,T]}$ be a real valued $\mathbb{H}$-adapted càdlàg process representing a reward, the payoff of a contract or a settlement that an agent receives at each time when he acts e.g. in a financial market. Given $\delta\in\mathcal{T}_T(\mathbb{H})$, the optimal stopping problem for the agent with reward ζ at time $\delta\in\mathcal{T}_T(\mathbb{H})$ consists in looking for a stopping time $\overline{\nu}\in\mathcal{T}_{\delta,T}(\mathbb{H})$ for which his conditional expected value $\mathbb{E}^{\mathbb{Q}}[\zeta_{\overline{\nu}}|\mathcal{H}_\delta]$ is maximal, i.e.

$$\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{\delta,T}(\mathbb{H})}\mathbb{E}^{\mathbb{Q}}[\zeta_\nu|\mathcal{H}_\delta] \quad=\quad \mathbb{E}^{\mathbb{Q}}[\zeta_{\overline{\nu}}|\mathcal{H}_\delta]. \tag{4.19}$$

The stopping time $\overline{\nu}$ is referred to as an optimal stopping time for the problem (4.19) at time δ. We will simply say that $\overline{\nu}$ is an optimal stopping time if $\delta = 0$. The main tool to address the optimal stopping problem with reward ζ is the Snell envelope V of ζ defined as follows.

Definition 4.3.2. *Let $\zeta \in \mathcal{S}^1_T(\mathbb{H}, \mathbb{Q})$. Then the Snell envelope V of ζ is the smallest càdlàg supermartingale dominating ζ, i.e. for each $t \in [0,T], V_t \geq \zeta_t$ $\mathbb{Q}$-a.s. It is given by*

$$V_\delta \quad = \quad \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{\delta,T}(\mathbb{H})} \mathbb{E}^{\mathbb{Q}}\left[\zeta_\nu | \mathcal{H}_\delta\right], \ \delta \in \mathcal{T}_T(\mathbb{H}). \tag{4.20}$$

Remark 4.3.3. *Let $\zeta^1, \zeta^2 \in \mathcal{S}^1_T(\mathbb{H}, \mathbb{Q})$ with respective Snell envelope V^1 and V^2. It follows from the explicit representation (4.20) that if $\zeta^1_t = \zeta^2_t, t \in [\nu, T]$, for some $\nu \in \mathcal{T}_T(\mathbb{H})$, then $V^1_t = V^2_t,\ t \in [\nu, T]$.*

Due to the explicit formula (4.20), V inherits the integrability properties of ζ.

Proposition 4.3.4. *Let $\zeta \in \mathcal{D}_T(\mathbb{H}, \mathbb{Q})$ and V its Snell envelope. Then $V \in \mathcal{D}_T(\mathbb{H}, \mathbb{Q})$. Moreover, if $\zeta \in \mathcal{S}^p_T(\mathbb{H}, \mathbb{Q})$ for some $p > 1$, then $V \in \mathcal{S}^p_T(\mathbb{H}, \mathbb{Q})$.*

Proof. The first statement follows from [EK81, Proposition 2.2.9]. For the second statement, we consider the martingale K defined by $K_t = \mathbb{E}\left[\sup_{0\leq s\leq T} |\zeta_s| | \mathcal{H}_t\right],\ t \in [0,T]$. Then $K \in \mathcal{S}^p_T(\mathbb{H}, \mathbb{Q})$ and $|V^p| \leq K^p$. Applying Doob's maximal inequality we obtain

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{0\leq t\leq T} |V_t|^p\right] \quad \leq \quad \mathbb{E}^{\mathbb{Q}}\left[\sup_{0\leq t\leq T}\left(\mathbb{E}^{\mathbb{Q}}\left[\sup_{0\leq s\leq T} |\zeta_s| | \mathcal{H}_t\right]\right)^p\right] = \mathbb{E}^{\mathbb{Q}}\left[\sup_{0\leq t\leq T} K^p_t\right] < +\infty.$$

□

The following theorem from [EK81] provides a necessary and sufficient condition for optimality. It highlights the crucial role played by the Snell envelope.

Theorem 4.3.5. *[EK81, Theorem 2.31] Let $r \geq 1$, $\zeta \in \mathcal{S}^r_T(\mathbb{H}, \mathbb{Q})$ and V the Snell envelope of ζ. Let $\delta \in \mathcal{T}_T(\mathbb{H})$. Then $\overline{\nu} \in \mathcal{T}_{\delta,T}(\mathbb{H})$ is an optimal stopping time for (4.19) if and only if the following two conditions are satisfied:*

1. *$V_{\overline{\nu}} = \zeta_{\overline{\nu}}$,*
2. *the stopped process $V_{\cdot\wedge\overline{\nu}}$ is an $\mathbb{H}$-martingale on $[\delta, \overline{\nu}]$.*

Assume that ζ is left-upper semicontinuous or V is regular. Then

$$\overline{\nu} \quad = \quad \inf\{t \geq \delta : V_t = \zeta_t\} \wedge T$$

is an optimal stopping time at time δ.

Remark 4.3.6. *An optimal stopping time $\overline{\nu}$ does not always exist (see [Ham02] or [KS98, Example D.11] for a counterexample) and if it exists, it might not be unique (see [KS98, Example D.14]).*

4.3.3 Problem formulation for optimal stopping problems

Now we consider the special case where $\mathbb{H} = \mathbb{G}$ and $\zeta \in \mathcal{S}^r_T(\mathbb{G}, \mathbb{P})$ for some $r \geq 1$. As ζ is càdlàg, there exists an $\mathbb{F}$-adapted càdlàg process ζ^{b} and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process ζ^{d} such that

$$\zeta_t \quad = \quad \zeta^{\mathrm{b}}_t 1_{\{t<\tau\}} + \zeta^{\mathrm{d}}_t(\tau) 1_{\{t\geq\tau\}},\ t \in [0,T]. \tag{4.21}$$

Our problem of interest is to look for a stopping time $\overline{\nu} \in \mathcal{T}_T(\mathbb{G})$ such that

$$\mathbb{E}\left[\zeta_{\overline{\nu}}\right] = \sup_{\nu \in \mathcal{T}_T(\mathbb{G})} \mathbb{E}\left[\zeta_\nu\right]. \tag{4.22}$$

As illustrated in Theorem 4.3.5, the Snell envelope V of ζ is the main tool for addressing optimality in (4.22). Since V is $\mathbb{G}$-optional, by Proposition 4.2.11 there exist an $\mathcal{O}(\mathbb{F})$-measurable process V^{b} and an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process V^{d} such that

$$V_t = V_t^{\mathrm{b}} 1_{\{t<\tau\}} + V_t^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}}, \; t \in [0, T]. \tag{4.23}$$

In view of Theorem 4.3.5 and the optional splitting formula for $\mathbb{G}$-stopping times, the explicit expressions of V^{b} and $V^{\mathrm{d}}(\tau)$ are necessary in order to provide a characterization of optimal stopping times both before and after τ. As V describes the maximal conditional value the agent can achieve at every time conditioned on the information he possesses, the knowledge of V^{b} and $V^{\mathrm{d}}(\tau)$ is important for the description of the actions of the agent before and after τ. Our decomposition approach to solve (4.22) will consist in identifying V^{b} and $V^{\mathrm{d}}(\tau)$ and their connections to optimal stopping problems in the filtrations $\mathbb{F}$ and $\mathbb{G}^\tau$.

4.3.4 Optional splitting formula for the Snell envelope

First let us provide some intuition on the candidate processes V^{b} and $V^{\mathrm{d}}(\tau)$. We recall that $D_t = 1_{\{t \geq \tau\}}, t \in \mathbb{R}^+$. We begin with $V^{\mathrm{d}}(\tau)$. By (4.21), ζ and $\zeta^{\mathrm{d}}(\tau) D$ are indistinguishable after τ. Thus by Remark 4.3.3 their respective Snell envelopes w.r.t. $\mathbb{G}$ are indistinguishable after τ. As $\mathbb{G}$ and $\mathbb{G}^\tau$ coincide after τ, the Snell envelope of $\zeta^{\mathrm{d}}(\tau) D$ w.r.t. $\mathbb{G}$ should be indistinguishable from its Snell envelope w.r.t. $\mathbb{G}^\tau$ after τ. Indeed after τ, the agent has full information on τ and will incorporate this information to obtain the best exercising stopping time. A candidate for $V^{\mathrm{d}}(\tau)$ after τ is thus given by

$$V_t^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}} = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{t,T}(\mathbb{G}^\tau)} \mathbb{E}\left[\zeta_\nu^{\mathrm{d}}(\tau) D_\nu \middle| \mathcal{G}_t^\tau\right] 1_{\{t \geq \tau\}}, \; t \in [0, T].$$

Up to τ, the information flow available to the agent is given by $\mathbb{F}$ [5]. Therefore the best approximation w.r.t. $\mathbb{F}$ of his maximal expected conditional value up to τ is given by J where, for $t \in [0, T]$

$$J_t = \mathbb{E}\left[V_{t\wedge\tau} | \mathcal{F}_t\right] = \mathbb{E}\left[V_t^{\mathrm{b}} 1_{\{t<\tau\}} + V_\tau^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}} | \mathcal{F}_t\right] = V_t^{\mathrm{b}} G_t + \int_0^t V_u^{\mathrm{d}}(u) \alpha_u^{\mathrm{d}}(u) \eta(du).$$

Since J is the best approximation of V based on the information $\mathbb{F}$, intuitively, J represents the dynamic value process of the stopping problem faced by the agent before τ. To obtain the corresponding reward, we note that as the information flow of the agent is given by $\mathbb{F}$, he is acting on the assumption that default has not occurred yet. Thus his reward should take into account the survival probability of τ and a compensation for the possibility of default. A choice for his reward is thus the process Υ given by

$$\Upsilon_t = \mathbb{E}\left[\zeta_t^{\mathrm{b}} 1_{\{t<\tau\}} + V_\tau^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}} | \mathcal{F}_t\right] = \zeta_t^{\mathrm{b}} G_t + \int_0^t V_u^{\mathrm{d}}(u) \alpha_u^{\mathrm{d}}(u) \eta(du), \; t \in [0, T].$$

The process J being the $\mathbb{F}$-Snell envelope of $\Upsilon = \zeta^{\mathrm{b}} G + \int_0^\cdot V_u^{\mathrm{d}}(u) \alpha_u^{\mathrm{d}}(u) \eta(du)$ entails that

$$V_t^{\mathrm{b}} G_t = \operatorname*{ess\,sup}_{\sigma^{\mathrm{b}} \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[\zeta_{\sigma^{\mathrm{b}}}^{\mathrm{b}} G_{\sigma^{\mathrm{b}}} + \int_t^{\sigma^{\mathrm{b}}} V_u^{\mathrm{d}}(u) \alpha_u^{\mathrm{d}}(u) \eta(du) \middle| \mathcal{F}_t\right], \; t \in [0, T].$$

[5] $\mathcal{F}_\tau = \mathcal{G}_\tau$ by Corollary 4.2.15

The following theorem justifies the above intuition for V^{b} and $V^{\mathrm{d}}(\tau)$. It also provides a necessary and sufficient condition for optimality before and after default. We use the terminology $\mathbb{H}$-Snell envelope and $\mathbb{H}$-supermartingale (resp. martingale) to emphasize the filtration. The reference probability is $\mathbb{P}$.

Theorem 4.3.7. *Let $r > 1$, $\zeta \in \mathcal{S}^r_T(\mathbb{G}, \mathbb{P})$ with optional splitting formula given by (4.21) and V its $\mathbb{G}$-Snell envelope. Let V^d be the $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process such that*

$$V^d_\delta(\tau) = \operatorname*{ess\,sup}_{\nu \in \mathcal{T}_{\delta,T}(\mathbb{G}^\tau)} \mathbb{E}\left[\zeta^d_\nu(\tau) D_\nu | \mathcal{G}^\tau_\delta\right], \ \delta \in \mathcal{T}_T(\mathbb{G}^\tau), \tag{4.24}$$

and J the $\mathbb{F}$-Snell envelope of $\Upsilon = \zeta^b G + \int_0^\cdot V^d_u(u)\alpha^d_u(u)\eta(du)$, i.e.

$$J_{\delta^b} = \operatorname*{ess\,sup}_{\nu^b \in \mathcal{T}_{\delta^b,T}(\mathbb{F})} \mathbb{E}\left[\zeta^b_{\nu^b} G_{\nu^b} + \int_0^{\nu^b} V^d_u(u)\alpha^d_u(u)\eta(du) \middle| \mathcal{F}_{\delta^b}\right], \ \delta^b \in \mathcal{T}_T(\mathbb{F}). \tag{4.25}$$

Then the following properties are satisfied:

a) V obeys the following optional splitting formula

$$V_t = V^b_t 1_{\{t<\tau\}} + V^d_t(\tau) 1_{\{t \geq \tau\}}, \ t \in [0,T], \tag{4.26}$$

where for $t \in [0,T]$

$$V^b_t G_t = J_t - \int_0^t V^d_u(u)\alpha^d_u(u)\eta(du). \tag{4.27}$$

Moreover we have

$$\sup_{\nu \in \mathcal{T}_T(\mathbb{G})} \mathbb{E}[\zeta_\nu] = \sup_{\nu^b \in \mathcal{T}_T(\mathbb{F})} \mathbb{E}\left[\zeta^b_{\nu^b} G_{\nu^b} + \int_0^{\nu^b} V^d_u(u)\alpha^d_u(u)\eta(du)\right] = \mathbb{E}\left[V^b_0\right]. \tag{4.28}$$

b) Let $\overline{\nu} = \overline{\nu}^b 1_{\{\overline{\nu}^b<\tau\}} + \overline{\nu}^d(\tau) 1_{\{\overline{\nu}^b \geq \tau\}} \in \mathcal{T}_T(\mathbb{G})$. Then $\overline{\nu}$ is an optimal stopping time for (4.22) if and only if the following two equalities hold:

$$\mathbb{E}\left[V^b_0\right] = E[V_0] = \mathbb{E}\left[\zeta^b_{\delta^b} G_{\overline{\nu}^b} + \int_0^{\overline{\nu}^b} V^d_u(u)\alpha^d_u \eta(du)\right], \tag{4.29}$$

$$\mathbb{E}\left[\zeta^d_{\overline{\nu}^d(\tau)}(\tau) 1_{\{\overline{\nu}^b \geq \tau\}} | \mathcal{G}^\tau_{T\wedge\tau}\right] = V^d_{T\wedge\tau}(\tau) 1_{\{\overline{\nu}^b \geq \tau\}} = V^d_\tau(\tau) 1_{\{\overline{\nu}^b \geq \tau\}}. \tag{4.30}$$

c) Assume that $\zeta^b G$ and $\zeta^d(\tau) D$ are left-upper semicontinuous. Let $\overline{\nu}^b$ and $\overline{\nu}^d(\tau)$ be defined as follows:

$$\overline{\nu}^b := \inf\left\{t \geq 0 : V^b_t = \zeta^b_t\right\} \wedge T \ \text{and} \ \overline{\nu}^d(\tau) := \inf\left\{t \geq T \wedge \tau : V^d_t(\tau) = \zeta^d_t(\tau)\right\} \wedge T.$$

Then $\overline{\nu} = \overline{\nu}^b 1_{\{\overline{\nu}^b<\tau\}} + (\tau \vee \overline{\nu}^d(\tau)) 1_{\{\overline{\nu}^b \geq \tau\}}$ is an optimal stopping time for (4.22).

Proof. a) First we check that V^{b} and $V^{\mathrm{d}}(\tau)$ are well defined. By Proposition 4.3.4, $V^{\mathrm{d}}(\tau) \in \mathcal{S}^r_T(\mathbb{G}^\tau, \mathbb{P})$ since $\zeta^{\mathrm{d}}(\tau) D \in \mathcal{S}^r_T(\mathbb{G}^\tau, \mathbb{P})$. As $V^{\mathrm{d}}(\tau) \in \mathcal{S}^r_T(\mathbb{G}^\tau, \mathbb{P})$ and $\zeta \in \mathcal{S}^r_T(\mathbb{G}, \mathbb{P})$, Lemma 4.2.24 entails that $\Upsilon = \zeta^{\mathrm{b}} G + \int_0^\cdot V^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du) \in \mathcal{S}^p_T(\mathbb{F}, \mathbb{P})$ for some $p > 1$. Hence $J \in \mathcal{S}^1_T(\mathbb{F}, \mathbb{P})$ and therefore $V^{\mathrm{b}} G$ is well defined and integrable. We verify in the following steps that V is the $\mathbb{G}$-Snell envelope of ζ.

Step 1. We show that V is a $\mathbb{G}$-supermartingale with càdlàg paths. As $V^{\mathrm{d}}(\tau)$ is the $\mathbb{G}^\tau$-Snell envelope of $\zeta^{\mathrm{d}}(\tau) D$, it is a $\mathbb{G}^\tau$-supermartingale. By Proposition 4.2.22, $\left\{V^{\mathrm{d}}_t(u)\alpha^{\mathrm{d}}_t(u), t \in [u,T]\right\}$

is an $\mathbb{F}$-supermartingale for η-almost all $u \in [0,T]$. Also, J is an $\mathbb{F}$-supermartingale as the $\mathbb{F}$-Snell envelope of Υ. V is therefore a $\mathbb{G}$-supermartingale by Proposition 4.2.23. The processes V^{b} and $V^{\mathrm{d}}(\tau)D$ being càdlàg, V is càdlàg.

Step 2. V dominates ζ. By the representations of V^{b} and J given respectively by (4.27) and (4.25), $V^{\mathrm{b}} \geq \zeta^{\mathrm{b}}$. The definition of $V^{\mathrm{d}}(\tau)$ implies that $V^{\mathrm{d}}(\tau)D \geq \zeta^{\mathrm{d}}(\tau)D$. Hence (4.26) yields $V \geq \zeta$.

Step 3. V is the smallest supermartingale that dominates ζ. Let Z be a supermartingale that dominates ζ with optional splitting formula $Z_t = Z^{\mathrm{b}}_t 1_{\{t<\tau\}} + Z^{\mathrm{d}}_t(\tau)1_{\{t\geq\tau\}}, t \in [0,T]$. Then $Z^{\mathrm{d}}(\tau)D$ dominates $\zeta^{\mathrm{d}}(\tau)D$ and $Z^{\mathrm{d}}(\tau)D$ is a $\mathbb{G}^\tau$-supermartingale. Since $V^{\mathrm{d}}(\tau)$ is the $\mathbb{G}^\tau$-Snell envelope of $\zeta^{\mathrm{d}}(\tau)D$ it follows from the minimality of the Snell envelope that for every $\nu \in \mathcal{T}_T(\mathbb{G}^\tau)$

$$V^{\mathrm{d}}_\nu(\tau)1_{\{\nu\geq\tau\}} \quad \leq \quad Z^{\mathrm{d}}_\nu(\tau)1_{\{\nu\geq\tau\}}. \tag{4.31}$$

In particular for $t \in [0,T]$

$$V^{\mathrm{d}}_t(\tau)1_{\{t\geq\tau\}} \quad \leq \quad Z^{\mathrm{d}}_t(\tau)1_{\{t\geq\tau\}},\ t \in [0,T].$$

It remains to show that $Z^{\mathrm{b}} \geq V^{\mathrm{b}}$ or equivalently $\Pi = Z^{\mathrm{b}}G + \int_0^\cdot V^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du) \geq J$ where J is given by (4.25). J being the $\mathbb{F}$-Snell envelope of Υ, it suffices to show that Π is an $\mathbb{F}$-supermartingale dominating Υ. We recall that as Z is a $\mathbb{G}$-supermartingale, the process $Z^{\mathrm{b}}G + \int_0^\cdot Z^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)$ is an $\mathbb{F}$-supermartingale by Proposition 4.2.23. Now using (4.31) with $\nu = T \wedge \tau$, we obtain

$$V^{\mathrm{d}}_\tau(\tau)1_{\{T\geq\tau\}} \leq Z^{\mathrm{d}}_\tau(\tau)1_{\{T\geq\tau\}}.$$

By Lemma 4.2.16, we infer that $Z^{\mathrm{d}}_u(u) \geq V^{\mathrm{d}}_u(u)$ $\mathbb{P}$-a.s. for η-almost all $u \in [0,T]$. The last inequality and the $\mathbb{F}$-supermartingale property of $Z^{\mathrm{b}}G + \int_0^\cdot Z^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)$ imply that Π is an $\mathbb{F}$-supermartingale. Hence $\Pi \geq J$. As $Z^{\mathrm{b}} \geq \zeta^{\mathrm{b}}, \Pi \geq \Upsilon$. Step 3 is proven.

By uniqueness of the Snell envelope, Steps 1, 2 and 3 show that V is the $\mathbb{G}$-Snell envelope of ζ.

The equality (4.28) follows from the equalities $\sup_{\nu\in\mathcal{T}_T(\mathbb{G})} \mathbb{E}[\zeta_\nu] = \mathbb{E}[V_0] = \mathbb{E}\left[V^{\mathrm{b}}_0\right]$ and the definition of V^{b}_0 given by (4.27).

b) $'' \Rightarrow''$. Assume that $\overline{\nu} = \overline{\nu}^{\mathrm{b}}1_{\{\overline{\nu}^{\mathrm{b}}<\tau\}} + \overline{\nu}^{\mathrm{d}}(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}} \in \mathcal{T}_T(\mathbb{G})$ is an optimal stopping time. Then by Theorem 4.3.5, $\zeta_{\overline{\nu}} = V_{\overline{\nu}}$ and $V_{\cdot\wedge\overline{\nu}}$ is a $\mathbb{G}$-martingale. Proposition 4.2.23 implies that the stopped process $J_{\cdot\wedge\overline{\nu}^{\mathrm{b}}}$ is an $\mathbb{F}$-martingale and

$$\mathbb{E}\left[V^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}}|\mathcal{G}^\tau_{T\wedge\tau}\right] \quad = \quad V^{\mathrm{d}}_\tau(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}}. \tag{4.32}$$

The equality $\zeta_{\overline{\nu}} = V_{\overline{\nu}}$ yields $\zeta^{\mathrm{b}}_{\overline{\nu}^{\mathrm{b}}} = V^{\mathrm{b}}_{\overline{\nu}^{\mathrm{b}}}$. This leads to $J_{\overline{\nu}^{\mathrm{b}}} = \zeta^{\mathrm{b}}_{\overline{\nu}^{\mathrm{b}}}G_{\overline{\nu}^{\mathrm{b}}} + \int_0^{\overline{\nu}^{\mathrm{b}}} V^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du) = \Upsilon_{\overline{\nu}^{\mathrm{b}}}$. J being the $\mathbb{F}$-Snell envelope of Υ and $J_{\cdot\wedge\overline{\nu}^{\mathrm{b}}}$ an $\mathbb{F}$-martingale, $\overline{\nu}^{\mathrm{b}}$ is an optimal stopping time for the problem (4.28) by Theorem 4.3.5. Consequently (4.29) holds by optimality of $\overline{\nu}^{\mathrm{b}}$. It follows from the equality $\zeta_{\overline{\nu}} = V_{\overline{\nu}}$ that $\zeta^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}} = V^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}}$ which together with (4.32) gives (4.30).

$'' \Leftarrow''$ By (4.29) and (4.30) we obtain

$$\begin{aligned}
\mathbb{E}[\zeta_{\overline{\nu}}] &= \mathbb{E}\left[\zeta_{\overline{\nu}^{\mathrm{b}}}1_{\{\overline{\nu}^{\mathrm{b}}<\tau\}} + \mathbb{E}\left[\zeta^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}}|\mathcal{G}^\tau_{T\wedge\tau}\right]\right] \\
&= \mathbb{E}\left[\zeta^{\mathrm{b}}_{\overline{\nu}^{\mathrm{b}}}1_{\{\overline{\nu}^{\mathrm{b}}<\tau\}} + V^{\mathrm{d}}_\tau(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}}\right] = \mathbb{E}\left[V^{\mathrm{b}}_0\right].
\end{aligned}$$

By (4.28), $\overline{\nu}$ is an optimal stopping time.

c) As G is strictly positive, $\overline{\nu}^{\mathrm{b}} = \inf\{t \geq 0 : J_t = \Upsilon_t\} \wedge T$. Since Υ is left-upper semicontinuous, $\overline{\nu}^{\mathrm{b}}$ satisfies (4.29). $\zeta^{\mathrm{d}}(\tau)D$ being left upper semicontinuous, $\overline{\nu}^{\mathrm{d}}(\tau)$ is an optimal stopping time for the stopping problem with reward $\zeta^{\mathrm{d}}(\tau)D$ at time $T \wedge \tau$. Hence

$V^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}(\tau) = \zeta^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}(\tau)D_{\overline{\nu}^{\mathrm{d}}(\tau)}$. The $\mathbb{G}^\tau$-martingale property of $V^{\mathrm{d}}(\tau)$ on $[T\wedge\tau, T]$ implies that $V^{\mathrm{d}}_{T\wedge\tau}(\tau) = \mathbb{E}\left[\zeta^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}(\tau)D_{\overline{\nu}^{\mathrm{d}}(\tau)}|\mathcal{G}^\tau_{T\wedge\tau}\right]$. Since $\{\overline{\nu}^{\mathrm{b}} \geq \tau\} \in \mathcal{G}^\tau_{T\wedge\tau}$, we have

$$V^{\mathrm{d}}_{T\wedge\tau}(\tau)1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}} \quad = \quad \mathbb{E}\left[\zeta^{\mathrm{d}}_{\overline{\nu}^{\mathrm{d}}(\tau)}1_{\{\overline{\nu}^{\mathrm{d}}(\tau)\geq\tau\}}1_{\{\overline{\nu}^{\mathrm{b}}\geq\tau\}}|\mathcal{G}^\tau_{T\wedge\tau}\right].$$

On the event $\{\overline{\nu}^{\mathrm{b}} \geq \tau\}, \overline{\nu}^{\mathrm{d}}(\tau) = \overline{\nu}^{\mathrm{d}}(\tau)\vee\tau$ and thus the above equality shows that $\overline{\nu}^{\mathrm{d}}(\tau)\vee\tau$ satisfies (4.30). We conclude from b) that $\overline{\nu}$ is an optimal stopping time. □

Remark 4.3.8. *Theorem 4.3.7 interpret the actions of the agent in the following way. From the characterization of an optimal stopping time given by Theorem 4.3.7 b) and the equality (4.28), it appears that to solve the optimal stopping problem (4.22) the agent has to proceed in two steps. In a first step, he begins by assessing his maximal conditional value conditioned on the event that default has occurred, i.e.* $V^d(\tau)$. *Having full knowledge of* τ, *he acts as an insider and* $V^d(\tau)$ *is identified as the maximal conditional value of a stopping problem in the filtration* $\mathbb{G}^\tau$, *see (4.24). In a second step, he addresses the optimal stopping problem (4.28) which yields the same value as the original stopping problem (4.22). The reward* $\Upsilon = \zeta^b G + \int_0^\cdot V^d_u(u)\alpha^d_u(u)\eta(du)$ *in (4.28) consists of a modification of the initial reward* ζ^b *which accounts only for paths for which default does not occur before* T *and a compensating factor which is the best approximation w.r.t.* $\mathbb{F}$ *of the maximal conditional value at default, i.e.* $V^d_\tau(\tau)$.

The following corollary shows that before default, an agent with the reward ζ faces the same optimal control as an agent with reward X where $X_t = \zeta^{\mathrm{b}}_t 1_{\{t<\tau\}} + V^{\mathrm{d}}_\tau(\tau)1_{\{t\geq\tau\}}, t\in[0,T]$.

Corollary 4.3.9. *We keep the notation and assumptions of Theorem 4.3.7. Assume that* $\zeta^d(\tau)D = C_\tau D$. *where* C *is an* $\mathcal{O}(\mathbb{F})$*-measurable process. Then for* $t\in[0,T]$

$$\begin{aligned} V^d_t(\tau)1_{\{t\geq\tau\}} &= C_\tau 1_{\{t\geq\tau\}}, \\ V^b_t G_t &= \operatorname*{ess\,sup}_{\nu^b\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}\left[\zeta^b_{\nu^b}G_{\nu^b} + \int_t^{\nu^b} C_u\alpha^d_u(u)\eta(du)\Big|\mathcal{F}_t\right]. \end{aligned}$$

The $(\mathbb{G},\mathbb{P})$*-Snell envelope of* X *where* $X_t = \zeta^b_t 1_{\{t<\tau\}} + V^d_\tau(\tau)1_{\{t\geq\tau\}},\ t\in[0,T]$, *is given by* $V_{\cdot\wedge\tau}$.

Proof. We show the first equality. By Theorem 4.3.7, $V^{\mathrm{d}}(\tau)$ is the $\mathbb{G}^\tau$-Snell envelope of $\zeta^{\mathrm{d}}(\tau)D$. Let $\delta\in\mathcal{T}_T(\mathbb{G}^\tau)$. Clearly the family $\left\{\mathbb{E}\left[\zeta^{\mathrm{d}}_\nu(\tau)D_\nu|\mathcal{G}^\tau_\delta\right],\ \nu\in\mathcal{T}_T(\mathbb{G}^\tau)\right\}$ is stable by supremum. Moreover, $\zeta^{\mathrm{d}}_\nu(\tau)D_\nu D_\delta = C_\tau D_\delta$ for $\nu\in\mathcal{T}_{\delta,T}(\mathbb{G}^\tau)$. Thus

$$V^{\mathrm{d}}_\delta(\tau)D_\delta = \left(\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{\delta,T}(\mathbb{G}^\tau)} \mathbb{E}\left[\zeta^{\mathrm{d}}_\nu(\tau)D_\nu|\mathcal{G}_\delta\right]\right)D_\delta = \left(\operatorname*{ess\,sup}_{\nu\in\mathcal{T}_{\nu,T}(\mathbb{G}^\tau)} \mathbb{E}\left[\zeta^{\mathrm{d}}_\nu(\tau)D_\nu D_\delta|\mathcal{G}_\delta\right]\right)D_\delta = C_\tau D_\delta.$$

This proves the first equality. Furthermore, taking $\delta = T\wedge\tau$ and applying Lemma 4.2.16, we obtain that $V^{\mathrm{d}}_u(u) = C_u$ $\mathbb{P}$-a.s. for η-almost all $u\in[0,T]$. The second equality follows from Theorem 4.3.7. The second assertion is an application of the first by taking $C = V^{\mathrm{d}}_\cdot(\cdot)$. □

Remark 4.3.10. *For* $\zeta^d(\tau)D = C_\tau D$, *the Snell envelope is uniquely determined by* V^b *and solving the optimal stopping problem (4.22) is equivalent to solving an optimal stopping problem in the filtration* $\mathbb{F}$ *namely, (4.28). The decomposition approach thus leads to a filtration reduction as termed in [BCJR09]. The particular case* $\zeta^d(\tau)D = C_\tau D$ *has also been treated in [Szi05] under the immersion hypothesis.*

In the sequel, we provide a recursive formula for V^{b}. This will be useful in the next chapter to identify the RBSDE satisfied by the pre-default value of the solution to a RBSDE in the filtration $\mathbb{G}$.

Theorem 4.3.11. *Let $r > 1$, $\zeta \in \mathcal{S}^r_T(\mathbb{G},\mathbb{P})$ with optional splitting formula (4.21) and V its $\mathbb{G}$-Snell envelope with optional splitting formula*

$$V_t = V^b_t 1_{\{t<\tau\}} + V^d_t(\tau)1_{\{t\geq\tau\}},\ t \in [0,T], \tag{4.33}$$

where V^b and $V^d(\tau)$ are given respectively by (4.24) and (4.27). Suppose that there exists $l > \frac{2r}{r-1}$ such that $\mathbb{E}\left[e^{l\int_0^T \lambda^{\mathbb{F}}_u \eta(du)}\right] < +\infty$ where $\lambda^{\mathbb{F}}$ is the $\mathbb{F}$-intensity of τ. Let $\widehat{\mathbb{Q}}$ be the equivalent probability measure to $\mathbb{P}$ on $(\Omega,\mathcal{A})$ with density on $\mathcal{F}_T$ given by

$$d\widehat{\mathbb{Q}}/d\mathbb{P}\big|_{\mathcal{F}_T} = L^{\mathbb{F}}_T = G_T e^{\int_0^T \lambda^{\mathbb{F}}_u \eta(du)}. \tag{4.34}$$

Then for every $\nu^b \in \mathcal{T}_T(\mathbb{F})$ we have

$$V^b_{\nu^b} \;=\; \operatorname*{ess\,sup}_{\sigma^b\in\mathcal{T}_{\nu^b,T}(\mathbb{F})} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\zeta^b_{\sigma^b} + \int_{\nu^b}^{\sigma^b}\left(V^d_u(u) - V^b_{u-}\right)\lambda^{\mathbb{F}}_u\eta(du)\Big|\mathcal{F}_{\nu^b}\right]. \tag{4.35}$$

Proof. First we show some integrability properties which result from the hypothesis on $\lambda^{\mathbb{F}}$. We recall that $L^{\mathbb{F}} = Ge^{\int \lambda^{\mathbb{F}}_u\eta(du)}$ is a $\mathbb{F}$-local martingale by Proposition 4.2.4. Now as $e^{\int_0^T \lambda^{\mathbb{F}}_u du}$ is integrable and $G \leq 1$, $L^{\mathbb{F}}$ is a uniformly integrable $\mathbb{F}$-martingale. By Theorem 4.3.7, $J = V^{\mathrm{b}}G + \int_0^\cdot V^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)$ is an $\mathbb{F}$-supermartingale as the $\mathbb{F}$-Snell envelope of $\Upsilon = \zeta^{\mathrm{b}}G + \int V^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du)$. Since $\zeta \in \mathcal{S}^r_T(\mathbb{G},\mathbb{P})$, Proposition 4.3.4 entails that $V \in \mathcal{S}^r_T(\mathbb{G},\mathbb{P})$ and we infer from Proposition 4.2.26 that $\Delta = V^{\mathrm{b}}L^{\mathbb{F}} + \int \left(V^{\mathrm{d}}_u(u) - V^{\mathrm{b}}_{u-}\right)L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)$ is an $\mathbb{F}$-supermartingale of class (D).
We proceed to show the equality (4.35). Let $\nu^{\mathrm{b}} \in \mathcal{T}(\mathbb{F})$.

"$\geq$". Δ is an $(\mathbb{F},\mathbb{P})$-supermartingale and $V^{\mathrm{b}} \geq \zeta^{\mathrm{b}}$. Consequently for $\sigma^{\mathrm{b}} \in \mathcal{T}_{\nu^{\mathrm{b}},T}(\mathbb{F})$, we have

$$V^{\mathrm{b}}_{\nu^{\mathrm{b}}}L^{\mathbb{F}}_{\nu^{\mathrm{b}}} \;\geq\; \mathbb{E}\left[\zeta^{\mathrm{b}}_{\sigma^{\mathrm{b}}}L^{\mathbb{F}}_{\sigma^{\mathrm{b}}} + \int_{\nu^{\mathrm{b}}}^{\sigma^{\mathrm{b}}}\left(V^{\mathrm{d}}_u(u) - V^{\mathrm{b}}_{u-}\right)L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)\Big|\mathcal{F}_{\nu^{\mathrm{b}}}\right].$$

Taking ess sup on both sides leads to the inequality

$$V^{\mathrm{b}}_{\nu^{\mathrm{b}}}L_{\nu^{\mathrm{b}}} \;\geq\; \operatorname*{ess\,sup}_{\sigma^{\mathrm{b}}\in\mathcal{T}_{\nu^{\mathrm{b}},T}(\mathbb{F})} \mathbb{E}\left[\zeta^{\mathrm{b}}_{\sigma^{\mathrm{b}}}L^{\mathbb{F}}_{\sigma^{\mathrm{b}}} + \int_{\nu^{\mathrm{b}}}^{\sigma^{\mathrm{b}}}\left(V^{\mathrm{d}}_u(u) - V^{\mathrm{b}}_{u-}\right)L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)\Big|\mathcal{F}_{\nu^{\mathrm{b}}}\right]. \tag{4.36}$$

Applying Bayes' formula, we see that "$\geq$" is satisfied.

"$\leq$" For $\epsilon > 0$, we consider $\nu^{\mathrm{b}}_\epsilon$ defined by $\nu^{\mathrm{b}}_\epsilon = \inf\{t \geq \nu^{\mathrm{b}} : J_t \leq \Upsilon_t + \epsilon\} \wedge T$. Clearly $\nu^{\mathrm{b}}_\epsilon \in \mathcal{T}_{\nu^{\mathrm{b}},T}(\mathbb{F})$ and by [LM84, Proposition 14], J is a $\mathbb{F}$-martingale on $[\nu^{\mathrm{b}},\nu^{\mathrm{b}}_\epsilon]$. We deduce from Proposition 4.2.26 that Δ is a uniformly integrable $\mathbb{F}$-martingale on $[\nu^{\mathrm{b}},\nu^{\mathrm{b}}_\epsilon]$. Hence

$$V^{\mathrm{b}}_{\nu^{\mathrm{b}}}L^{\mathbb{F}}_{\nu^{\mathrm{b}}} \;=\; \mathbb{E}\left[V^{\mathrm{b}}_{\nu^{\mathrm{b}}_\epsilon}L^{\mathbb{F}}_{\nu^{\mathrm{b}}_\epsilon} + \int_{\nu^{\mathrm{b}}}^{\nu^{\mathrm{b}}_\epsilon}\left(V^{\mathrm{d}}_u(u) - V^{\mathrm{b}}_{u-}\right)L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)\Big|\mathcal{F}_{\nu^{\mathrm{b}}}\right]. \tag{4.37}$$

From the relation $J = V^{\mathrm{b}}G + \int_0^\cdot V^{\mathrm{d}}_u(u)\alpha^{\mathrm{d}}_u(u)\eta(du), L^{\mathbb{F}} = Ge^{\int_0^\cdot \lambda^{\mathbb{F}}_u\eta(du)}$ and the definition of $\nu^{\mathrm{b}}_\epsilon$, we have

$$V^{\mathrm{b}}_{\nu^{\mathrm{b}}_\epsilon}L^{\mathbb{F}}_{\nu^{\mathrm{b}}_\epsilon} \;\leq\; \zeta^{\mathrm{b}}_{\nu^{\mathrm{b}}_\epsilon}L_{\nu^{\mathrm{b}}_\epsilon} + \epsilon e^{\int_0^T \lambda^{\mathbb{F}}_u\eta(du)}.$$

Consequently from (4.37), we infer that

$$\begin{aligned} V^{\mathrm{b}}_{\nu^{\mathrm{b}}}L^{\mathbb{F}}_{\nu^{\mathrm{b}}} &\leq \mathbb{E}\left[\zeta^{\mathrm{b}}_{\nu^{\mathrm{b}}_\epsilon}L^{\mathbb{F}}_{\nu^{\mathrm{b}}_\epsilon} + \int_{\nu^{\mathrm{b}}}^{\nu^{\mathrm{b}}_\epsilon}\left(V^{\mathrm{d}}_u(u) - V^{\mathrm{b}}_{u-}\right)L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)\Big|\mathcal{F}_{\nu^{\mathrm{b}}}\right] + \epsilon\mathbb{E}\left[e^{\int_0^T \lambda^{\mathbb{F}}_u\eta(du)}|\mathcal{F}_{\nu^{\mathrm{b}}}\right], \\ &\leq \operatorname*{ess\,sup}_{\sigma^{\mathrm{b}}\in\mathcal{T}_{\nu^{\mathrm{b}},T}(\mathbb{F})} \mathbb{E}\left[\zeta^{\mathrm{b}}_{\sigma^{\mathrm{b}}}L^{\mathbb{F}}_{\sigma^{\mathrm{b}}} + \int_{\nu^{\mathrm{b}}}^{\nu^{\mathrm{b}}}\left(V^{\mathrm{d}}_u(u) - V^{\mathrm{b}}_{u-}\right)L^{\mathbb{F}}_u\lambda^{\mathbb{F}}_u\eta(du)\Big|\mathcal{F}_{\nu^{\mathrm{b}}}\right] + \epsilon\mathbb{E}\left[e^{\int_0^T \lambda^{\mathbb{F}}_u\eta(du)}|\mathcal{F}_{\nu^{\mathrm{b}}}\right]. \end{aligned}$$

Taking the limit as ϵ goes to 0, one obtains

$$V^{\mathrm{b}}_{\nu^{\mathrm{b}}} G_{\nu^{\mathrm{b}}} \quad \leq \quad \operatorname*{ess\,sup}_{\sigma^{\mathrm{b}} \in \mathcal{T}_{\nu^{\mathrm{b}},T}(\mathbb{F})} \mathbb{E}\left[\zeta^{\mathrm{b}}_{\sigma^{\mathrm{b}}} L^{\mathbb{F}}_{\sigma^{\mathrm{b}}} + \int_{\nu^{\mathrm{b}}}^{\nu^{\mathrm{b}}} \left(V^{\mathrm{d}}_u(u) - V^{\mathrm{b}}_{u-} \right) L^{\mathbb{F}}_u \lambda^{\mathbb{F}}_u \eta(du) \middle| \mathcal{F}_{\nu^{\mathrm{b}}} \right].$$

An application of Bayes' formula yields $'' \leq''$. Therefore (4.35) holds. □

Remark 4.3.12. *The sole purpose of the integrability restriction on $\lambda^{\mathbb{F}}$ is to guarantee that Δ is of class (D). One can therefore replace the restriction in Theorem 4.3.11 by the conditions: Δ is of class (D) and $e^{\int_0^T \lambda^{\mathbb{F}}_u \eta(du)} \in L^1(\Omega, \mathcal{F}_T)$.*

4.4 Hedging of defaultable claims of American type

The aim of this section is to use the two step decomposition approach for optimal stopping problems to construct hedging strategies for American contingent claims in a financial market with information flow $\mathbb{G}$. A particular advantage of this approach is the understanding of the actions of the agents involved as it gives the precise description of hedges against such claims both before and after default.

4.4.1 Financial market

First we fix some terminology. $T \in (0, +\infty)$ is the trading time horizon. Let $n \in \mathbb{N}$ and $\mathbb{H} = (\mathcal{H}_t)_{t \in [0,T]}$ be a filtration in $\mathcal{A}$ satisfying the usual conditions. Let S be an $\mathbb{R}^n$-valued $\mathbb{H}$-semimartingale modeling the dynamics of traded assets. We will refer to $(S, \mathbb{H})$ as the financial market with the price process of traded assets modeled by S and information flow $\mathbb{H}$.

Definition 4.4.1. *We call $\pi = (\pi^i)_{1 \leq i \leq n}$ a trading strategy in the market $(S, \mathbb{H})$ if π is $\mathcal{P}(\mathbb{H})$-measurable and the stochastic integral $\int_0^\cdot \pi dS$ is well defined. For $i = 1, \cdots, n$, π^i denotes the number of shares held in the asset S^i. We denote by $\mathcal{L}(S, \mathbb{H})$ the set of trading strategies. To a portfolio (x, π) where $x > 0$ is the initial capital and $\pi \in \mathcal{L}(S, \mathbb{H})$, we associate the wealth process $X^{\pi,x}$ given by*

$$X^{\pi,x}_t \quad = \quad x + \int_0^t \pi_u dS_u, \;\; t \in [0, T].$$

We recall the following definition pertaining to the characterization of non-arbitrage in the sense of *No Free Lunch with Vanishing Risk* (NFLVR) and market completeness (see [DS94]).

Definition 4.4.2. *A probability measure $\mathbb{Q}$ on $(\Omega, \mathcal{A})$ is an $\mathbb{H}$-equivalent martingale measure (e.m.m.) for S if $\mathbb{Q}$ is equivalent to $\mathbb{P}$ on $\mathcal{A}$ and S is an $(\mathbb{H}, \mathbb{Q})$-local martingale. $\mathcal{M}^e(S, \mathbb{H})$ denotes the set of $\mathbb{H}$-e.m.m. for S.*
The market $(S, \mathbb{H})$ is said to be complete if $\mathcal{M}^e(S, \mathbb{H})$ is reduced to a singleton. If $\mathcal{M}^e(S, \mathbb{H})$ has more than one element, the market $(S, \mathbb{H})$ is referred to as an incomplete market.

Default free asset

Our primary asset is default free and modeled by an $\mathbb{F}$-semimartingale S^1. We refer to $(S^1, \mathbb{F})$ as the default free market. As S^1 is also $\mathbb{G}$ and $\mathbb{G}^\tau$-adapted, the markets $(S^1, \mathbb{G})$ and $(S^1, \mathbb{G}^\tau)$ are well defined. They represent respectively the view of S^1 from the perspective of an agent monitoring the default time τ and one possessing the full information on τ. We assume throughout that :

Assumption 4.4.3. *There exists a probability measure $\mathbb{Q}^{\mathbb{F}}$ on $(\Omega, \mathcal{A})$ such that $\mathcal{M}^e(S^1, \mathbb{F}) = \left\{ \mathbb{Q}^{\mathbb{F}} \right\}$.*

We set $d\mathbb{Q}^{\mathbb{F}}/d\mathbb{P}|_{\mathcal{F}_T} = Z_T^{\mathbb{F}}$ where $Z^{\mathbb{F}}$ is defined by

$$Z_t^{\mathbb{F}} = \mathbb{E}\left[Z_T^{\mathbb{F}}|\mathcal{F}_t\right], \ t \in [0,T]. \tag{4.38}$$

Let $\mathbb{Q}^{\mathbb{G}^\tau}$ be the probability measure on $(\Omega, \mathcal{A})$ equivalent to $\mathbb{P}$ and satisfying

$$\mathbb{E}\left[d\mathbb{Q}^{\mathbb{G}^\tau}/d\mathbb{P}|\mathcal{G}_t^\tau\right] \ = \ Z_t^{\mathbb{F}}/\alpha_t^{\mathrm{d}}(\tau) = Z_t^{\mathbb{G}^\tau}, \ t \in [0,T]. \tag{4.39}$$

Remark 4.4.4. *Assumption 4.4.3 has the following consequences for the markets $(S^1, \mathbb{G})$ and $(S^1, \mathbb{G}^\tau)$:*

(i) The completeness of the market $(S^1, \mathbb{F})$ entails that $\mathbb{F}$ admits a martingale representation theorem w.r.t. S^1 under $\mathbb{Q}^{\mathbb{F}}$, see [HWY92, Theorem 13.9]. It follows from [Ame00, GP98, CJZ13] that $\mathbb{G}^\tau$ admits a martingale representation theorem under the martingale preserving measure $\mathbb{Q}^{\mathbb{G}^\tau}$. Thus $(S^1, \mathbb{G}^\tau)$ is also complete and we have $\mathcal{M}^e(S^1, \mathbb{G}^\tau) = \left\{\mathbb{Q}^{\mathbb{G}^\tau}\right\}$.

(ii) Under the measure $\mathbb{Q}^{\mathbb{G}^\tau}$, $\mathbb{F}$ and $\sigma(\tau)$ are independent, see [Ame00, Theorem 3.1]. Thus the immersion hypothesis holds under $\mathbb{Q}^{\mathbb{G}^\tau}$ and it follows from [CJN12, Corollary 4.6] that $\mathcal{M}^e(S, \mathbb{G}) \neq \emptyset$. Furthermore for every $\mathbb{Q} \in \mathcal{M}^e(S^1, \mathbb{G})$, the immersion hypothesis holds under $\mathbb{Q}$ and $\mathbb{Q}|\mathcal{F}_T = \mathbb{Q}^{\mathbb{F}}$,see [BSJ04, JLC09a].

The market $(S^1, \mathbb{G}^\tau)$ is complete by Remark 4.4.4. However, this is not the case for $(S^1, \mathbb{G})$. The following proposition characterizes the Radon-Nikodym densities of the equivalent local martingale measures for the market $(S^1, \mathbb{G})$.

Proposition 4.4.5. *Let $\mathbb{Q}$ be a probability measure on $(\Omega, \mathcal{A})$ with*

$$Q_t = \mathbb{E}\left[d\mathbb{Q}/d\mathbb{P}|\mathcal{G}_t\right] \ = \ Q_t^b 1_{\{t<\tau\}} + Q_t^d(\tau)1_{\{t\geq\tau\}}, \ t \in [0,T]. \tag{4.40}$$

i) The density hypothesis holds under $\mathbb{Q}$ and the $(\mathbb{F}, \mathbb{Q})$-density process $\alpha^{d,\mathbb{Q}}$ is given by

$$\alpha_t^{d,\mathbb{Q}}(u) \ = \ \begin{cases} \dfrac{Q_t^d(u)\alpha_t^d(u)}{\mathbb{E}\left[Q_t\middle|\mathcal{F}_t\right]}, \ t \in [u,T], u \geq 0, \\[2ex] \dfrac{\mathbb{E}\left[Q_u^d(u)\alpha_u^d(u)\middle|\mathcal{F}_t\right]}{\mathbb{E}\left[Q_t\middle|\mathcal{F}_t\right]}, \ t < u, u \geq 0. \end{cases} \tag{4.41}$$

ii) $\mathbb{Q} \in \mathcal{M}^e(S^1, \mathbb{G})$ if and only if

$$Q_t^d(u) \ = \ \frac{1}{\alpha_t^d(u)}\frac{Q_u^d(u)\alpha_u^d(u)}{Z_u^{\mathbb{F}}}Z_t^{\mathbb{F}}, \ t \in [u,T], u \geq 0, \tag{4.42}$$

$$Q_t^b G_t \ = \ Z_t^{\mathbb{F}}\left(1 - \int_0^t \frac{Q_u^d(u)\alpha_u^d(u)}{Z_u^{\mathbb{F}}}\eta(du)\right), \ t \in [0,T]. \tag{4.43}$$

iii) Assume that $\mathbb{Q} \in \mathcal{M}^e(S^1, \mathbb{G})$. Then for $H \in L^1(\mathcal{G}_T, \mathbb{Q})$ and $t \in [0,T]$

$$\mathbb{E}^{\mathbb{Q}}\left[H|\mathcal{G}_t\right]1_{\{t\geq\tau\}} = \mathbb{E}^{\mathbb{Q}}\left[H|\mathcal{G}_t^\tau\right]1_{\{t\geq\tau\}} \ = \ \mathbb{E}^{\mathbb{Q}^{\mathbb{G}^\tau}}\left[H|\mathcal{G}_t^\tau\right]1_{\{t\geq\tau\}}. \tag{4.44}$$

Proof. i) See Theorem 6.1 in [EKJJ10].
ii) $'' \Rightarrow''$: By Remark 4.4.4 ii), $\mathbb{F}$ is immersed in $\mathbb{G}$ and $\mathbb{E}\left[Q_t|\mathcal{F}_t\right] = Z_t^{\mathbb{F}}, \ t \in [0,T]$. As τ avoids $\mathbb{F}$-stopping times, we infer from [JLC09b, Corollary 3.1] that $\alpha_t^{\mathrm{d},\mathbb{Q}}(u) = \alpha_u^{d,\mathbb{Q}}(u), \ t \geq u, u \in \mathbb{R}^+$,

which leads to (4.42).
By the density hypothesis and (4.42), the equality $\mathbb{E}\left[Q_t|\mathcal{F}_t\right] = Z_t^{\mathbb{F}},\ t \in [0,T]$, is equivalent to :

$$Z_t^{\mathbb{F}} \;=\; Q_t^{\mathrm{b}} G_t + \int_0^t Q_t^{\mathrm{d}}(u)\alpha_t^{\mathrm{d}}(u)\eta(du) = Q_t^{\mathrm{b}} G_t + Z_t^{\mathbb{F}} \int_0^t \frac{Q_u^{\mathrm{d}}(u)\alpha_u^{\mathrm{d}}(u)}{Z_u^{\mathbb{F}}}\eta(du),\ t \in [0,T].$$

We obtain (4.43) by rearranging terms.
$'' \Leftarrow''$: Using (4.42) and (4.43) we have that $\mathbb{E}\left[Q_t|\mathcal{F}_t\right] = Z_t^{\mathbb{F}}$ for $t \in [0,T]$ and $\alpha_t^{d,\mathbb{Q}}(u) = \frac{Q_u^{\mathrm{d}}(u)\alpha_u^{\mathrm{d}}(u)}{Z_u^{\mathbb{F}}}$ for $t \in [u,T]$. Hence $\alpha^{d,\mathbb{Q}}(u)$ is constant after u and consequently $\mathbb{F}$ is immersed in $\mathbb{G}$ under $\mathbb{Q}$ since τ avoids $\mathbb{F}$-stopping times. To show that $\mathbb{Q} \in \mathcal{M}^e(S^1,\mathbb{G})$, it suffices to show that S^1 is an $(\mathbb{F},\mathbb{Q})$-local martingale. Let $(\sigma_n)_{n\in\mathbb{N}} \in \mathcal{T}_T(\mathbb{F})$ be a localizing sequence for the $(\mathbb{F},\mathbb{Q}^{\mathbb{F}})$-local martingale S^1. By the tower property, we have for $s,t \in [0,T]$ with $s \le t$ and $n \in \mathbb{N}$

$$\begin{aligned}
\mathbb{E}\left[S^1_{\sigma_n\wedge t}Q_t|\mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}\left[Q_t S^1_{t\wedge\sigma_n}|\mathcal{F}_t\right]|\mathcal{F}_s\right] \\
&= \mathbb{E}\left[S^1_{t\wedge\sigma_n}\mathbb{E}\left[Q_t|\mathcal{F}_t\right]|\mathcal{F}_s\right] = \mathbb{E}\left[S^1_{t\wedge\sigma_n}Z_t^{\mathbb{F}}|\mathcal{F}_s\right].
\end{aligned}$$

Given that for every $n \in \mathbb{N}$, the stopped process $S^1_{\cdot\wedge\sigma_n}$ is an $(\mathbb{F},\mathbb{Q}^{\mathbb{F}})$-martingale, it follows from the above equalities that for $s,t \in [0,T]$ with $s \le t$ and $n \in \mathbb{N}$, we have

$$\mathbb{E}\left[S^1_{\sigma_n\wedge t}Q_t|\mathcal{F}_s\right] \;=\; \mathbb{E}\left[S^1_{t\wedge\sigma_n}Z_t^{\mathbb{F}}|\mathcal{F}_s\right] = S^1_{s\wedge\sigma_n}Z_s^{\mathbb{F}} = S^1_{s\wedge\sigma_n}\mathbb{E}\left[Q_s|\mathcal{F}_s\right].$$

We deduce that S^1 is an $(\mathbb{F},\mathbb{Q})$-local martingale.
iii) Let $\mathbb{Q} \in \mathcal{M}^e(S^1,\mathbb{G})$ and $X \in L^1(\mathcal{G}_T,\mathbb{Q})$ and $t \in [0,T]$. Applying Proposition 4.2.18 we have $\mathbb{E}^{\mathbb{Q}}\left[H|\mathcal{G}_t\right]1_{\{t\ge\tau\}} = \mathbb{E}^{\mathbb{Q}}\left[H|\mathcal{G}_t^\tau\right]1_{\{t\ge\tau\}}$. For the second equality, we apply successively Bayes' formula, Proposition 4.2.18 and the formulas (4.42) and (4.39) which give

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[H|\mathcal{G}_t\right]1_{\{t\ge\tau\}} &= \frac{\mathbb{E}\left[Q_T H|\mathcal{G}_t\right]}{Q_t}1_{\{t\ge\tau\}} = \frac{\mathbb{E}\left[Q_T H|\mathcal{G}_t^\tau\right]}{Q_t}1_{\{t\ge\tau\}} = \frac{\mathbb{E}\left[Z_T^{\mathbb{G}^\tau}H|\mathcal{G}_t^\tau\right]}{Z_t^{\mathbb{G}^\tau}}1_{\{t\ge\tau\}} \\
&= \mathbb{E}^{\mathbb{Q}^{\mathbb{G}^\tau}}\left[H|\mathcal{G}_t^\tau\right]1_{\{t\ge\tau\}}.
\end{aligned}$$

□

Remark 4.4.6. *a) By (4.42) and (4.43), a probability measure $\mathbb{Q} \in \mathcal{M}^e(S^1,\mathbb{G})$ with density process given by (4.40) is determined uniquely by the trace process $\left\{Q_u^d(u),\ u \ge 0\right\}$. The only constraints on $Q_\cdot^d(\cdot)$ are strict positivity and $\mathbb{E}\left[\int_0^{+\infty} Q_u^d(u)\alpha_u^d(u)\eta(du)\right] = \mathbb{Q}(\tau < +\infty) = 1$. Clearly there are infinitely many choices for $Q_\cdot^d(\cdot)$. The market $(S,\mathbb{G})$ is therefore incomplete.*

b) For any $\mathcal{G}_T^\tau$-measurable random variable H, (4.44) shows that $H1_{\{T\ge\tau\}}$ has a unique price in the market $(S,\mathbb{G})$ given by $\mathbb{E}^{\mathbb{Q}^{\mathbb{G}^\tau}}\left[H1_{\{T\ge\tau\}}\right]$. This is to be expected as $\mathbb{G}$ and $\mathbb{G}^\tau$ coincide after τ.

Defautable zero coupon bond

Our second financial asset is the defaultable zero coupon bond (DZC) with maturity T (i.e. a claim which pays one unit at time T if and only if τ has not occurred before T) which we assume to be traded in the market $(S^1,\mathbb{G})$ with price process $\rho(t,T),\ t \in [0,T]$. We recall that $\rho(T,T) = 1_{\{T<\tau\}}$ and $D_t = 1_{\{\tau\le t\}},\ t \in [0,T]$. Our market of interest consists of S^1 and ρ and we assume the following :

Assumption 4.4.7. *The market $(S^1, \rho, \mathbb{G})$ is arbitrage free, i.e. $\mathcal{M}^e(S^1, \rho, \mathbb{G}) \neq \emptyset$.*

In view of the incompleteness of the market $(S^1, \mathbb{G})$, defaultable sensitive contingent claims such as $H1_{\{T \leq \tau\}}$ where H is an $\mathcal{F}_T$-measurable random variable cannot be hedged with the asset S^1 alone given a predefined price $x > 0$. As highlighted in [JLC09a, Section 3.2.1] one needs an asset sensitive to the jump risk induced by the default time τ in order to hedge defaultable claims. This is the role played by the DZC. We introduce some tools necessary to describe the dynamics of $\rho(\cdot, T)$. Due to Assumption 4.4.7 there exists $\mathbb{Q} \in \mathcal{M}^e(S^1, \mathbb{G})$ such that $\rho(\cdot, T)$ is a $\mathbb{Q}$-local martingale. We suppose that $\mathbb{Q}$ is given and fixed. Using the density hypothesis under $\mathbb{Q}$ and the fact that $\mathbb{Q}|_{\mathcal{F}_T} = \mathbb{Q}^{\mathbb{F}}$, we have for $t \in [0, T]$

$$\rho(t, T) \quad = \quad \mathbb{E}^{\mathbb{Q}}\left[1_{\{T<\tau\}}|\mathcal{G}_t\right] = (1/G_t^{\mathbb{Q}})\mathbb{E}^{\mathbb{Q}}\left[G_t^{\mathbb{Q}}|\mathcal{F}_t\right] 1_{\{t<\tau\}} = J_t m_t, \tag{4.45}$$

where $G^{\mathbb{Q}}$ is the survival process of τ w.r.t. $\mathbb{Q}$ and

$$J \quad = \quad (1/G^{\mathbb{Q}})(1 - D), \quad m_t = \mathbb{E}^{\mathbb{Q}^{\mathbb{F}}}\left[G_T^{\mathbb{Q}}|\mathcal{F}_t\right], \ t \in [0, T].$$

Let $\lambda^{\mathbb{F},\mathbb{Q}}$ be the $\mathbb{F}$-intensity of τ w.r.t. $\mathbb{Q}$. Since the immersion hypothesis holds under $\mathbb{Q}$, for every $u \geq 0$, the martingale $\alpha^{d,\mathbb{Q}}(u)$ is constant after u (see [JLC09b, Corollary 3.1]) and thus the survival process $G^{\mathbb{Q}}$ is increasing and given by

$$G_t^{\mathbb{Q}} = e^{-\int_0^t \lambda_s^{\mathbb{F},\mathbb{Q}} \eta(ds)}, \ t \in [0, T].$$

Clearly $G^{\mathbb{Q}}$ is continuous as η is non-atomic. The compensator $\Lambda^{\mathbb{G},\mathbb{Q}}$ of D w.r.t. $\mathbb{Q}$ is given by

$$\Lambda_t^{\mathbb{G},\mathbb{Q}} = \int_0^t \lambda_s^{\mathbb{F},\mathbb{Q}} 1_{\{s<\tau\}} \eta(ds), \ t \in [0, T].$$

Let $N^{\mathbb{Q}}$ be the $(\mathbb{G}, \mathbb{Q})$-local martingale defined by

$$N^{\mathbb{Q}} = D - \Lambda^{\mathbb{G},\mathbb{Q}}.$$

Due to the completeness of $(S^1, \mathbb{F})$, there exists $\pi^{\mathbb{Q}} \in \mathcal{L}(S^1, \mathbb{F})$ such that

$$dm_t \quad = \quad \pi_t^{\mathbb{Q}} dS_t^1, \ t \in [0, T].$$

Note that by Proposition 5.1 in [EKJJ10] J is a $(\mathbb{G}, \mathbb{Q})$-local martingale. Its dynamics is described by

$$dJ_t = -J_{t-} dN_t^{\mathbb{Q}}, \ t \in [0, T]. \tag{4.46}$$

An application of Itô's formula to the product $\rho = J \cdot m$ yields the following dynamics of ρ.

Lemma 4.4.8. *Suppose that Assumptions 4.4.3 and 4.4.7 hold. Then $\rho(\cdot, T)$ satisfies*

$$d\rho(t, T) \quad = \quad C_t dS_t^1 - \rho(t-, T) dN_t^{\mathbb{Q}}, t \in [0, T], \tag{4.47}$$

where $C_t = J_{t-}\pi_t^{\mathbb{Q}}, t \in [0, T]$.

Remark 4.4.9. *As $\mathbb{Q} = \mathbb{Q}^{\mathbb{F}}$ on $\mathcal{F}_T$, $\mathbb{F}$ satisfies the martingale representation property w.r.t. S^1 under the measure $\mathbb{Q}$ and thus $\mathbb{G}$ admits a martingale representation theorem w.r.t. S^1 and $N^{\mathbb{Q}}$ under the measure $\mathbb{Q}$, see [CJZ13]. In case S^1 is continuous and has quadratic variation that is absolutely continuous w.r.t. to the Lebesgue measure, the market $(S^1, \rho, \mathbb{G})$ is complete (see [JLC09a, Theorem 3.1]).*

4.4.2 Hedging of American defaultable claims

We now consider the financial market $(S^1, \rho, \mathbb{G})$ with e.m.m. $\mathbb{Q}$ and an American contingent claim (ACC) defined by a nonnegative process $\zeta \in \mathcal{S}^1_T(\mathbb{G}, \mathbb{Q})$. From classical results (see [Ben84, Kar88, KK98, Kra96]), a no-arbitrage price of the ACC ζ is given by the value of the optimal stopping problem with reward ζ under the measure $\mathbb{Q}$. Our goal in this section is to employ the two step methodology to solve stopping problems in $\mathbb{G}$ to provide the explicit structure of the pre-default and post-default values of the hedging strategy π of the claim ζ. This leads to a description of the actions of agents in order to hedge such claims. We recall the following notion of hedge and arbitrage for the seller.

Definition 4.4.10. *An admissible portfolio (π, x) is a hedge against ζ if $X^{\pi,x}_t \geq \zeta_t,\ t \in [0, T]$. Let $x > 0$. x is an arbitrage price for an ACC ζ if there exists an admissible portfolio (u, π) with $u < x$ such that $X^{\pi,u} \geq \zeta$.*

Remark 4.4.11. *One defines analogously an ACC and hedges in the markets $(S^1, \mathbb{F})$ and $(S^1, \mathbb{G}^\tau)$.*

To achieve our goal, we introduce maps which give hedges against ACC in the complete markets $(S^1, \mathbb{G}^\tau)$ and $(S^1, \mathbb{F})$. Now for $\Theta \in \mathcal{S}^1_T(\mathbb{F}, \mathbb{Q}^{\mathbb{F}})$, let $S(\Theta)$ be its $(\mathbb{F}, \mathbb{Q}^{\mathbb{F}})$-Snell envelope and $S(\Theta) = S_0(\Theta) + M^{S(\Theta)} - K^{S(\Theta)}$ its Doob-Meyer's decomposition where $M^{S(\Theta)}$ is an $(\mathbb{F}, \mathbb{Q}^{\mathbb{F}})$-uniformly integrable martingale and $K^{S(\Theta)}$ an $\mathbb{F}$-predictable increasing process. The market $(S^1, \mathbb{F})$ being complete, there exists $\pi^{S(\Theta)} \in \mathcal{L}(S^1, \mathbb{F})$ such that

$$M^{S(\Theta)} \quad = \quad \int_0^{\cdot} \pi^{S(\Theta)} dS^1. \tag{4.48}$$

It is clear that $(S_0(\Theta), \pi^{S(\Theta)})$ is a hedge against Θ in the market $(S^1, \mathbb{F})$. Indeed,

$$X^{\pi^{S(\Theta)}, S_0(\Theta)} = S_0(\Theta) + M^{S(\Theta)} = S(\Theta) + K^{S(\Theta)} \geq \Theta.$$

We define the map $DFM_{(S^1,\mathbb{F})}$ yielding hedges in the default free market $(S^1, \mathbb{F})$ as follows:

$$DFM_{(S^1,\mathbb{F})}: \quad \mathcal{S}^1_T(\mathbb{F}, \mathbb{Q}^{\mathbb{F}}) \to \mathcal{L}(S^1, \mathbb{F}) \tag{4.49}$$
$$\Theta \mapsto \pi^{S(\Theta)}. \tag{4.50}$$

We will define the map $IM_{(S^1,\mathbb{G}^\tau)}$ yielding hedges in the insider market $(S^1, \mathbb{G}^\tau)$ in a similar way. For $\vartheta \in \mathcal{S}^1_T(\mathbb{G}^\tau, \mathbb{Q}^{\mathbb{G}^\tau})$, let $S(\vartheta)$ be its $(\mathbb{G}^\tau, \mathbb{Q}^{\mathbb{G}^\tau})$-Snell envelope and $S(\vartheta) = S_0(\vartheta) + M^{S(\vartheta)} - K^{S(\vartheta)}$ its Doob-Meyer's decomposition. The completeness of the market $(S^1, \mathbb{G}^\tau)$ entails that there exists $\pi^{S(\vartheta)} \in \mathcal{L}(S^1, \mathbb{G}^\tau)$ such that

$$M^{S(\vartheta)} \quad = \quad \int_0^{\cdot} \pi^{S(\vartheta)} dS^1.$$

It is clear that $(S_0(\vartheta), \pi^{S(\vartheta)})$ is a hedge against ϑ in the market $(S^1, \mathbb{G}^\tau)$. $IM_{(S^1,\mathbb{G}^\tau,p)}$ is then defined as follows

$$IM_{(S^1,\mathbb{G}^\tau)}: \quad \mathcal{S}^1_T(\mathbb{G}^\tau, \mathbb{Q}^{\mathbb{G}^\tau}) \to \mathcal{L}(S^1, \mathbb{G}^\tau) \tag{4.51}$$
$$\vartheta \mapsto \pi^{S(\vartheta)}. \tag{4.52}$$

Let $r > 1$. We consider an ACC $\zeta \in \mathcal{S}^r_T(\mathbb{G})$ with optional splitting formula

$$\zeta_t \quad = \quad \zeta^{\mathrm{b}}_t 1_{\{t<\tau\}} + \zeta^{\mathrm{d}}_t(\tau) 1_{\{t\geq\tau\}},\ t \in [0, T]. \tag{4.53}$$

Let V be the $(\mathbb{G}, \mathbb{Q})$-Snell envelope of ζ. The optional splitting formula of V is given in the following proposition:

Proposition 4.4.12. *Suppose that Assumptions 4.4.3 and 4.4.7 hold. Let V^d be an $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process and V^b the $\mathbb{F}$-adapted càdlàg process such that for every $t \in [0,T]$*

$$\begin{aligned} V_t^d(\tau) &= \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{t,T}(\mathbb{G}^\tau)} \mathbb{E}^{\mathbb{Q}^{\mathbb{G}^\tau}} \left[\zeta_\sigma^d(\tau) 1_{\{\sigma \geq \tau\}} | \mathcal{G}_t^\tau \right], \\ V_t^b G_t^{\mathbb{Q}} &= \operatorname*{ess\,sup}_{\sigma^b \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}^{\mathbb{F}}} \left[\zeta_{\sigma^b}^b G_{\sigma^b}^{\mathbb{Q}} + \int_t^{\sigma^b} V_u^d(u) \alpha_u^{d,\mathbb{Q}}(u) \eta(du) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Then V satisfies for $t \in [0,T]$

$$V_t = V_t^b 1_{\{t < \tau\}} + V_t^d(\tau) 1_{\{t \geq \tau\}}. \tag{4.54}$$

Proof. By Theorem 4.3.7, $V_t = \overline{V}_t^{\mathrm{b}} 1_{\{t<\tau\}} + \overline{V}_t^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}}$, $t \in [0,T]$ where $\overline{V}^{\mathrm{d}}$ (resp. $\overline{V}^{\mathrm{b}}$) is the $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$ (resp. $\mathcal{O}(\mathbb{F})$)-measurable process satisfying for $t \in [0,T]$

$$\begin{aligned} \overline{V}_t^{\mathrm{d}}(\tau) &= \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{t,T}(\mathbb{G}^\tau)} \mathbb{E}^{\mathbb{Q}} \left[\zeta_\sigma^{\mathrm{d}}(\tau) D_\sigma | \mathcal{G}_t^\tau \right] \text{ and} \\ \overline{V}_t^{\mathrm{b}} G_t^{\mathbb{Q}} &= \operatorname*{ess\,sup}_{\nu^{\mathrm{b}} \in \mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}^{\mathbb{Q}} \left[\zeta_{\nu^{\mathrm{b}}}^{\mathrm{b}} G_{\nu^{\mathrm{b}}} + \int_t^{\nu^{\mathrm{b}}} \overline{V}_u^{\mathrm{d}}(u) \alpha_u^{d,\mathbb{Q}}(u) \eta(du) \middle| \mathcal{F}_t \right]. \end{aligned}$$

Let $t \in [0,T]$. Then for $\sigma \in \mathcal{T}_{t,T}(\mathbb{G}^\tau)$, $\zeta_\sigma^{\mathrm{d}}(\tau) D_\sigma \in L^1(\mathcal{G}_T, \mathbb{Q})$, and Proposition 4.4.5 entails that $\mathbb{E}^{\mathbb{Q}} \left[\zeta_\sigma^{\mathrm{d}}(\tau) D_\sigma | \mathcal{G}_T^\tau \right] 1_{\{t \geq \tau\}} = \mathbb{E}^{\mathbb{Q}^{\mathbb{G}^\tau}} \left[\zeta_\sigma^{\mathrm{d}}(\tau) D_\sigma | \mathcal{G}_t^\tau \right] 1_{\{t \geq \tau\}}$. The family $\{\mathbb{E}^{\mathbb{Q}^{\mathbb{G}^\tau}} \left[\zeta_\sigma^{\mathrm{d}}(\tau) D_\sigma | \mathcal{G}_t^\tau \right], \sigma \in \mathcal{T}_{t,T}(\mathbb{G})\}$ being stable by pairwise maximization, we deduce from the definitions of $V^{\mathrm{d}}(\tau)$ and $\overline{V}^{\mathrm{d}}(\tau)$ that

$$V_t^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}} = \overline{V}_t^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}}.$$

Since $V^{\mathrm{d}}(\tau) D$ and $\overline{V}^{\mathrm{d}}(\tau) D$ are càdlàg, we infer from the above equality that they are indistinguishable. Moreover,

$$V_\tau^{\mathrm{d}}(\tau) 1_{\{T \geq \tau\}} = V_{T \wedge \tau}^{\mathrm{d}}(\tau) D_{T \wedge \tau} = \overline{V}_{T \wedge \tau}^{\mathrm{d}}(\tau) D_{T \wedge \tau} = \overline{V}_\tau^{\mathrm{d}}(\tau) 1_{\{T \geq \tau\}}.$$

Lemma 4.2.16 implies that $V_u^{\mathrm{d}}(u) = \overline{V}_u^{\mathrm{d}}(u)$ for η-almost all $u \in [0,T]$. Inserting the latter equation into the definition of $\overline{V}^{\mathrm{b}}$ and using the fact that $\mathbb{Q} = \mathbb{Q}^{\mathbb{F}}$, we deduce that $V^{\mathrm{b}} = \overline{V}^{\mathrm{b}}$. Hence (4.54) holds. □

Note that $V^{\mathrm{b}} G^{\mathbb{Q}} + \int_0^\cdot V_u^{\mathrm{d}}(u) \alpha_u^{d,\mathbb{Q}}(u) \eta(du)$ is the $(\mathbb{F}, \mathbb{Q}^{\mathbb{F}})$-Snell envelope of $\Upsilon \in \mathcal{S}_T^1(\mathbb{F}, \mathbb{Q}^{\mathbb{F}})$ defined by

$$\Upsilon = \zeta^{\mathrm{b}} G^{\mathbb{Q}} + \int_0^\cdot V_u^{\mathrm{d}}(u) \alpha_u^{d,\mathbb{Q}}(u) \eta(du). \tag{4.55}$$

A hedge against the ACC Υ in the market $(S^1, \mathbb{F})$ is given by $(V_0^{\mathrm{b}}, Z^{\mathrm{b}})$ where

$$Z^{\mathrm{b}} = DFM_{(S^1, \mathbb{F})}(\Upsilon). \tag{4.56}$$

A hedge against the ACC $\zeta^{\mathrm{d}}(\tau) D$ is given by $(V_0^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau))$ where $Z^{\mathrm{d}}(\tau)$ is defined by

$$Z^{\mathrm{d}}(\tau) = IM_{(S^1, \mathbb{G}^\tau)}(\zeta^{\mathrm{d}}(\tau) D). \tag{4.57}$$

The following theorem gives a hedge against ζ in the market $(S^1, \rho, \mathbb{G})$.

Theorem 4.4.13. *Suppose that Assumptions 4.4.3 and 4.4.7 hold. Suppose additionally that* $V_\cdot^d(\cdot) \in \mathcal{L}(S^1, \mathbb{G})$. *Let* Z^b *be given by (4.56) and* $Z^d(\tau)$ *by (4.57). We consider* $\pi = (Z, \Pi)$ *defined by*

$$\Pi_t = -\frac{V_t^d(t) - V_{t-}^b}{\rho(t-, T)} 1_{\{t \leq \tau\}}, \ t \in [0, T], \tag{4.58}$$

$$Z_t = C_t \Pi_t + \frac{1}{G_t^{\mathbb{Q}}} Z_t^b 1_{\{t \leq \tau\}} + Z_t^d(\tau) 1_{\{t > \tau\}}, \ t \in [0, T]. \tag{4.59}$$

A hedge against the ACC ζ *is given by* (V_0, π).

Proof. Let $V = V_0 + M^V - K^V$ be the Doob-Meyer decomposition of V with M^V its martingale part and K^V its finite variation part. For the proof of the assertion, it will suffice to show that

$$M^V = \int_0^\cdot Z_s dS_s^1 + \int_0^\cdot \Pi_s d\rho(s, T), \tag{4.60}$$

since $X^{\pi, V_0} = V_0 + M^V = V + K^V \geq V \geq \zeta$. To achieve this, we employ Itô's formula and the optional splitting formula (4.54).

We begin with $V^b(1 - D)$. We recall that $V^b G^{\mathbb{Q}} + \int_0^\cdot V_u^d(u) \alpha_u^{d,\mathbb{Q}}(u) \eta(du)$ is the $(\mathbb{F}, \mathbb{Q}^{\mathbb{F}})$-Snell envelope of Υ. By construction of Z^b, there exists an $\mathbb{F}$-predictable increasing process K^b such that

$$d(V^b G_t^{\mathbb{Q}}) = Z_t^b dS_t^1 - V_t^d(t) \alpha_t^{d,\mathbb{Q}}(t) \eta(dt) - dK_t^b, \ t \in [0, T].$$

Using the equality $V^b(1 - D) = V^b G^{\mathbb{Q}} J$ where J satisfies (4.46), we infer from the equation above that

$$d(V_t^b G_t^{\mathbb{Q}} J_t) = J_{t-} Z_t^b dS_t^1 - V_{t-}^b G_{t-}^{\mathbb{Q}} J_{t-} dN_t^{\mathbb{Q}} - J_{t-} V_t^d(t) \alpha_t^{d,\mathbb{Q}}(t) \eta(dt) - J_{t-} dK_t^b, \ t \in [0, T].$$

Next we derive an equation for $V^d(\tau) D$. By construction of $Z^d(\tau)$, there exists a $\mathbb{G}^\tau$-predictable increasing process $K^d(\tau)$ such that

$$dV_t^d(\tau) = Z_t^d(\tau) dS_t^1 - dK_t^d(\tau), \ t \in [0, T].$$

D being a pure jump semimartingale, we infer from [Pro04] that

$$[V^d(\tau), D]_t = \Delta_\tau V^d(\tau) D_t = \int_0^t (V_s^d(s) - V_{s-}^d(s)) dD_s = \int_0^t (V_s^d(s) - V_{s-}^d(\tau)) dD_s, \ t \in [0, T].$$

One obtains from Itô's formula that for $t \in [0, T]$

$$d(V_t^d(\tau) D_t) = D_{t-} dV_t^d(\tau) + V_{t-}^d(\tau) dD_t + [V^d(\tau), D]_t = D_{t-} Z_t^d(\tau) dS_t^1 + V_t^d(t) dD_t - D_{t-} dK_t^d(\tau).$$

By Proposition 4.4.5 iii), we have $\mathbb{Q} = \mathbb{Q}^{\mathbb{G}^\tau}$ on $\{t \geq \tau\}$ for every $t \in [0, T]$. It follows from Proposition 4.4.12 that

$$V_t^d(\tau) D_t = \left(\operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{t,T}(\mathbb{G}^\tau)} \mathbb{E}^{\mathbb{Q}} \left[\zeta_\sigma^d(\tau) D_\sigma | \mathcal{G}_t^\tau \right] \right) D_t, \ t \in [0, T].$$

We infer from Lemma 4.3.4 that $V^d(\tau) D \in \mathcal{S}_T^r(\mathbb{G}, \mathbb{Q})$ and therefore $V^d(\tau) D$ is a $(\mathbb{G}, \mathbb{Q})$-special semimartingale. Since S^1 is a $(\mathbb{G}, \mathbb{Q})$-local martingale and $K^d(\tau)$ is $\mathbb{G}^\tau$-predictable, we deduce that $\int_0^\cdot D_{s-} Z_s^d(\tau) dS_s^1 - \int_0^\cdot D_{s-} dK_s^d(\tau)$ is a $(\mathbb{G}, \mathbb{Q})$ special semimartingale. The equation for $V^d(\tau) D$ implies that $\int_0^\cdot V_s^d(s) dD_s$ is a $(\mathbb{G}, \mathbb{Q})$ special semimartingale. Relying on the decomposition $D = N^{\mathbb{Q}} - \Lambda^{\mathbb{G}, \mathbb{Q}}$, Theorem 2 in [CMS80] entails that

$$\int_0^t V_s^d(s) dD_s = \int_0^t V_s^d(s) dN_s^{\mathbb{Q}} + \int_0^t V_s^d(s) \lambda_s^{\mathbb{F}, \mathbb{Q}} (1 - D_s) \eta(ds), \ t \in [0, T]. \tag{4.61}$$

As $N^{\mathbb{Q}} = N^{\mathbb{Q}}_{\cdot\wedge\tau}$ and η is non-atomic, the above equality takes the form

$$\int_0^t V^{\mathrm{d}}_s(s)dD_s \quad = \quad \int_0^t V^{\mathrm{d}}_s(s)(1-D_{s-})dN^{\mathbb{Q}}_s + \int_0^t V^{\mathrm{d}}_s(s)\lambda^{\mathbb{F},\mathbb{Q}}_s(1-D_{s-})\eta(ds),\ t\in[0,T].$$

Taking this into account, for $t\in[0,T]$ the dynamical behavior of $V^{\mathrm{d}}(\tau)D$ can be rewritten in the form

$$d(V^{\mathrm{d}}_t(\tau)D_t) = D_{t-}Z^{\mathrm{d}}_t(\tau)dS^1_t + V^{\mathrm{d}}_t(t)(1-D_{t-})dN^{\mathbb{Q}}_t + V^{\mathrm{d}}_t(t)\lambda^{\mathbb{F},\mathbb{Q}}_t(1-D_{t-})\eta(dt) - D_{t-}dK^{\mathrm{d}}_t(\tau).$$

Using the equalities $G^{\mathbb{Q}}_{t-}J_{t-} = 1-D_{t-}$ and $J_{t-}\alpha^{d,\mathbb{Q}}_t(t) = \lambda^{\mathbb{F},\mathbb{Q}}_t(1-D_{t-}),\ t\in[0,T]$, and combining the equations for $V^{\mathrm{b}}G^{\mathbb{Q}}J$ and $V^{\mathrm{d}}(\tau)D$, one obtains the following equation for V

$$dV_t \quad = \quad (J_{t-}Z^{\mathrm{b}}_t + D_{t-}Z^{\mathrm{d}}_t(\tau))dS^1_t + (V^{\mathrm{d}}_t(t) - V^{\mathrm{b}}_{t-})(1-D_{t-})dN^{\mathbb{Q}}_t - dK^V_t,\ t\in[0,T],$$

where $K^V_t = \int_0^t J_{s-}dK^{\mathrm{b}}_s + \int_0^t D_{s-}dK^{\mathrm{d}}_s,\ t\in[0,T]$. Clearly K is $\mathbb{G}$-predictable and increasing. Since ρ vanishes after τ and $N^{\mathbb{Q}} = N^{\mathbb{Q}}_{\cdot\wedge\tau}$, we deduce from (4.47) that

$$dN^{\mathbb{Q}}_t \quad = \quad \frac{C_t}{\rho(t-,T)}dS^1_t - \frac{1}{\rho(t-,T)}d\rho(t,T),\ t\in[0,T].$$

Inserting the equation for $N^{\mathbb{Q}}$ in the equation describing the dynamics of V, we obtain that the martingale part M^V of V is given by (4.60). Hence (V_0, Z, Π) is a hedge against ζ. $\square$

Remark 4.4.14. *Theorem 4.4.13 establishes a two step approach to hedge the claim ζ based on the decomposition approach for solving the stopping problem with reward ζ. In a first step one acts as an insider possessing the full knowledge of τ and construct a hedge $(V^d_0(\tau), Z^d(\tau))$ against the claim $\zeta^d(\tau)D$ in the market $(S^1,\mathbb{G}^\tau)$. This is achieved by solving the optimal stopping problem with reward $\zeta^d(\tau)D$. In a second step, one addresses the optimal stopping problem with reward Υ and obtains a hedge (V^b_0, Z^b) against the ACC Υ in the market $(S^1,\mathbb{F})$. As a result, one has a precise knowledge of the pre-default and post-default values of π.*

Remark 4.4.15. *i) The trace process $V^d_\cdot(\cdot)$ is $\mathbb{F}$-optional. The $\mathbb{F}$-predictability of $V^d_\cdot(\cdot)$ is necessary for the definition of the stochastic integral $\int_0^\cdot V^d_\cdot(\cdot)dD$ and the decomposition (4.61). It also ensures that Π is predictable. It is satisfied if $\mathcal{O}(\mathbb{F}) = \mathcal{P}(\mathbb{F})$ or $\zeta^d(\tau)D = h_\tau D$ with $\mathbb{F}$-predictable h.*

ii) Even if $V^d_\cdot(\cdot)$ is $\mathbb{F}$-predictable, we may not have $\Pi\in\mathcal{L}(S^1,\mathbb{G})$ though $\Pi\in\mathcal{L}(N^{\mathbb{Q}},\mathbb{G})$. The assumption $V^d_\cdot(\cdot)\in\mathcal{L}(S^1,\mathbb{G})$ guarantees that $\Pi\in\mathcal{L}(S^1,\mathbb{G})\cap\mathcal{L}(\rho,\mathbb{G})$ since V^b_- and $\rho(-,T)$ are left-continuous and therefore locally bounded. This satisfied if for example $\zeta^d(\tau)D = h_\tau D$ where h is some $\mathbb{F}$-predictable bounded process.

iii) Theorem 4.4.13 generalizes the hedging results obtained [BSJ04] for defaultable claims of European type to defaultable claims of American type. Moreover the results in [BSJ04] are restricted to events before default.

5. Reflected BSDEs in a progressively enlarged filtration: a two step decomposition approach

5.1 Introduction

In this chapter, we address the solvability of reflected backward stochastic differential equations (RBSDEs for short). These are backward stochastic differential equations (BSDEs for short) for which the solution is required to stay above a certain process called the *barrier*. We consider the setup where our underlying filtration $\mathbb{G}$ is given by the *progressive enlargement* of a reference filtration $\mathbb{F}$ with a random time τ. We assume the density hypothesis on the regular conditional distribution of τ given $\mathbb{F}$ which guarantees that every $\mathbb{G}$-optional process satisfies the *optional splitting formula*, i.e. it can be identified with an $\mathbb{F}$-optional process before τ and a $\mathbb{G}^\tau$-optional process after τ, where $\mathbb{G}^\tau$ denotes the *initial enlargement* of $\mathbb{F}$ with τ. Our main goal in this chapter is to obtain existence of solutions for RBSDEs in the filtration $\mathbb{G}$ and this by providing their optional splitting formula. The precise formulations are given in Sections 5.2.3.

To achieve our goal, we establish in our main result Theorem 5.3.5 a two-step decomposition approach to solve RBSDEs in the filtration $\mathbb{G}$ which consists in solving two alternative weakly coupled systems of RBSDEs: first a RBSDE in the filtration $\mathbb{G}^\tau$ whose solution constitutes the value of the solution of the original RBSDE after τ, and then a RBSDE in the filtration $\mathbb{F}$ whose solution represents the value of the original RBSDE before τ. The second RBSDE in the filtration $\mathbb{F}$ is w.r.t. an auxiliary probability measure equivalent to the reference measure, and the coefficients of the RBSDE depend on the solution to the first RBSDE in the filtration $\mathbb{G}^\tau$. The global solution is then obtained by suitably concatenating the solutions resulting from the alternative RBSDEs. To identify the alternative RBSDEs, we make use of the one-to-one correspondence between solutions to RBSDEs and their Snell envelope representation [EKKP+97, LX05, Kli15] and the optional splitting formula of Snell envelopes established in Theorems 4.3.7 and 4.3.11. Besides identifying the system of RBSDEs, a key issue to address in our procedure is the stability of stochastic integration as we deal with different filtrations. We solve this by establishing a formula for stochastic integration w.r.t. $\mathbb{G}$-martingales.

Based on our decomposition approach, we provide new existence results for two classes of drivers. For the class of Lipschitz drivers and for a general filtration $\mathbb{F}$, we show the existence of L^p-solutions in Theorem 5.3.13. We also consider drivers of quadratic growth in the control variable and exponential growth in the jump variable. For this class, we restrict ourselves to the case where $\mathbb{F}$ is the completion of the filtration generated by a Brownian motion. Under the assumption that the input data of the RBSDEs are bounded, we show in Theorem 5.4.24 that a bounded solution exists. Under the additional assumption that the driver is locally Lipschitz in its variables, we provide *a priori* estimates of solutions and establish a comparison principle in Theorems 5.4.15 and 5.4.25, respectively. We note that our decomposition approach can also be applied to solve BSDEs in the filtration $\mathbb{G}$. In this context, a similar decomposition approach appears already in [ABSEL10, KL12] in the particular case where $\mathbb{F}$ is a Brownian filtration.

The rest of this chapter is structured as follows. In Section 5.2, we recall the definitions of progressive and initial enlargements as well as the main hypotheses that we consider. We also give the formulation of RBSDEs as well as the various concepts of solutions that we consider. We present our decomposition approach as well as its application to the solvability of RBSDEs in the case of Lipschitz drivers in Section 5.3. Section 5.4 is devoted to the solvability of RBSDEs of quadratic growth.

5.2 Mathematical framework and problem formulation

Throughout this chapter, $T \in (0, +\infty)$ is fixed and we work with a complete probability space $(\Omega, \mathcal{A}, \mathbb{P})$. We keep the same notation as in Chapter 4.

5.2.1 Filtration enlargements

Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a filtration of sub-σ-algebras of $\mathcal{A}$. We assume that $\mathcal{F}_0$ is complete and that $\mathbb{F}$ is right continuous. Let $\tau : \Omega \to \mathbb{R}^+$ be a non-negative and finite random time. We consider D the right-continuous indicator process of τ, i.e.

$$D_t = 1_{\{\tau \leq t\}},\ t \in \mathbb{R}^+.$$

To $\mathbb{F}$ and τ, we associate the enlarged filtrations $\mathbb{G}^\tau = (\mathcal{G}_t^\tau)_{t\geq 0}$ and $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, where

$$\mathcal{G}_t^\tau = \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma(\tau)\right),\ \mathcal{G}_t = \bigcap_{s>t} \left(\mathcal{F}_s \vee \sigma\left(D_u, u \leq s\right)\right), t \in \mathbb{R}^+. \tag{5.1}$$

The filtration $\mathbb{G}^\tau$ is known as the initial enlargement of $\mathbb{F}$ by τ while $\mathbb{G}$ is the progressive enlargement of $\mathbb{F}$ by τ. Throughout this chapter, we will employ the so called *density hypothesis* which guarantees the preservation of the semimartingale property in passing from $\mathbb{F}$ to $\mathbb{H} \in \{\mathbb{G}, \mathbb{G}^\tau\}$:

Assumption 5.2.1. *For any $t \in \mathbb{R}^+$, the conditional distribution of τ given $\mathcal{F}_t$ is equivalent to the law of τ denoted by η, i.e. there exists $\alpha_t^d : \Omega \times \mathbb{R}^+ \ni (\omega, u) \mapsto \alpha_t^d(\omega, u), \mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^+)$-measurable and strictly positive such that*

$$\mathbb{P}\left[\tau \in du | \mathcal{F}_t\right] = \alpha_t^d(\cdot, u)\eta(du)\ \mathbb{P}\text{-}a.s.$$

From [Jac85, Lemme 1.8] and [Ame00, Lemma 2.2] there exists $\alpha^{\mathrm{d}} : \mathbb{R}^+ \times \Omega \times \mathbb{R}^+ \to (0, +\infty)$ with the following properties

- the map $(t, \omega, u) \mapsto \alpha_t^{\mathrm{d}}(\omega, u)$ is $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+) - \mathcal{B}(\mathbb{R}^+)$ measurable,
- for all $u \in \mathbb{R}^+$, $\alpha^{\mathrm{d}}(\cdot, u)$ is a càdlàg $(\mathbb{F}, \mathbb{P})$-martingale,
- for every $t \in \mathbb{R}^+$, the measure $\alpha_t^{\mathrm{d}}(\cdot, u)\mathbb{P}\left(\tau \in du\right)$ on $\left(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+)\right)$ is a version of the conditional distribution $\mathbb{P}\left(\tau \in du \middle| \mathcal{F}_t\right), \mathbb{P}$-a.s.

The process α^{d} is referred to as the density process of τ w.r.t. $\mathbb{F}$ under the reference measure $\mathbb{P}$. **Notation**: Let $n \in \mathbb{N}$. For $X^{\mathrm{d}} : \Omega \times \mathbb{R}^+ \to \mathbb{R}^n$, $u \in \mathbb{R}^+$, we denote by $X^{\mathrm{d}}(u)$ (resp. $X^{\mathrm{d}}(\tau)$) the map $\Omega \ni \omega \mapsto X(\omega, u)$ (resp. $X^{\mathrm{d}}(\omega, \tau(\omega))$). Similarly for a process $X^{\mathrm{d}} : \mathbb{R}^+ \times \Omega \times \mathbb{R}^+ \to \mathbb{R}^n$ and $u \in \mathbb{R}^+$, we denote by $X^{\mathrm{d}}(u)$ (resp. $X^{\mathrm{d}}(\tau)$) the map $\mathbb{R}^+ \times \Omega \ni (t, \omega) \mapsto X_t^{\mathrm{d}}(\omega, u)$ (resp. $X_t^{\mathrm{d}}(\omega, \tau(\omega))$). We use a similar notation for processes defined on $[0, T]$. We recall some measurability properties from the previous chapter.

Proposition 5.2.2. *The following assertions are true:*

i) *$Y : [0, T] \times \Omega \to \mathbb{R}$ is $\mathcal{P}(\mathbb{G}^\tau)$(resp. $\mathcal{O}(\mathbb{G}^\tau)$)-measurable if and only if there exists a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$(resp. $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$)-measurable random variable $Y^d : [0, T] \times \Omega \times \mathbb{R}^+ \to \mathbb{R}$ such that*

$$Y = Y^d(\tau).$$

ii) *$K : [0, T] \times \Omega \to \mathbb{R}$ is $\mathcal{P}(\mathbb{G})$-measurable if and only if there exists a $\mathcal{P}(\mathbb{F})$-measurable random variable K^b and a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable random variable K^d such that for every $t \in [0, T]$*

$$K_t = K^b 1_{\{t \leq \tau\}} + K_t^d(\tau) 1_{\{t > \tau\}}. \tag{5.2}$$

iii) $K : [0,T] \times \Omega \to \mathbb{R}$ *is* $\mathcal{O}(\mathbb{G})$*-measurable if and only if there exists an* $\mathcal{O}(\mathbb{F})$*-measurable random variable* K^b *and an* $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$*-measurable random variable* K^d *such that for every* $t \in [0,T]$

$$K_t = K^b 1_{\{t<\tau\}} + K^d_t(\tau) 1_{\{t\geq\tau\}}. \tag{5.3}$$

In a nutshell, $\mathbb{G}^\tau$-predictable (resp. optional processes) can be viewed as $\mathbb{F}$-predictable (resp. optional) processes parametrized by the occurrence of τ. Note that the inclusion $\mathbb{F} \subseteq \mathbb{G} \subseteq \mathbb{G}^\tau$ together with the decomposition (5.2) show that one can identify a $\mathbb{G}$-predictable process before and after τ respectively with an $\mathbb{F}$-predictable process and a $\mathbb{G}^\tau$-predictable process. The decomposition of $\mathbb{G}$-optional processes given by (5.3) is known as the *optional splitting formula*, see [Son14]. For an $\mathcal{O}(\mathbb{G})$-measurable process K satisfying (5.3), we refer to K^{b} as the pre-default value of K and to $K^{\mathrm{d}}(\tau)$ as the post-default value. We employ a similar terminology for $\mathbb{G}$-predictable processes.

Throughout this work, we also make the following standing assumption on η, the law of τ:

Assumption 5.2.3. *η is absolutely continuous w.r.t. the Lebesgue measure on $\mathbb{R}^+$, i.e. there exists a positive Borel function $a^{\mathbb{F}}$ such that $\int_0^{+\infty} a^{\mathbb{F}}(u)du = 1$ and $\eta(du) = a^{\mathbb{F}}(u)du$ for every $u \in \mathbb{R}^+$.*

Assumption 5.2.3 entails that the law η of τ is non-atomic. Consequently, τ avoids $\mathbb{F}$ stopping times i.e., $\mathbb{P}(\tau = \nu) = 0,\ \forall\, \nu \in \mathcal{T}(\mathbb{F})$, see [EKJJ10, Corollary 2.2]. Note that the default process D is a submartingale and thus admits a compensator $\Lambda^{\mathbb{G}}$. To provide the exact form of the compensator, let us recall the conditional survival process G of τ defined for $t \in \mathbb{R}^+$ by

$$G_t \;=\; \mathbb{P}\left[\tau > t|\mathcal{F}_t\right] = \int_t^{+\infty} \alpha^{\mathrm{d}}_t(u)\eta(du) = 1 - \int_0^t \alpha^{\mathrm{d}}_t(u)\eta(du) = \mathbb{E}\left[\int_t^\infty \alpha^{\mathrm{d}}_u(u)\eta(du)|\mathcal{F}_t\right] \tag{5.4}$$

Since α^{d} is strictly positive and η is non-atomic, G is strictly positive. In [EKJJ10] the authors provide an additive and multiplicative decomposition of G which we now recall.

Proposition 5.2.4. *The Doob-Meyer decomposition of the survival process $(G_t)_{t\geq 0}$ is given by $G_t = 1 + M^{\mathbb{F}} - \int_0^t \alpha^d_u(u)\eta(du), t \geq 0,$ where $M^{\mathbb{F}}$ is the square integrable martingale given by*

$$M^{\mathbb{F}}_t \;=\; \mathbb{E}\left[\int_0^\infty \alpha^d_u(u)\eta(du)\Big|\mathcal{F}_t\right] - 1,\ t \in \mathbb{R}^+.$$

Let $\lambda^{\mathbb{F}}$ be the process defined by $\lambda^{\mathbb{F}}_t = \frac{\alpha^d_t(t)}{G_t},\ t \in \mathbb{R}^+$, and $L^{\mathbb{F}}$ the local martingale satisfying

$$dL^{\mathbb{F}}_t \;=\; e^{\int_0^t \lambda^{\mathbb{F}}_s \eta(ds)} dM^{\mathbb{F}}_t,\ L^{\mathbb{F}}_0 = 1,\ t \geq 0. \tag{5.5}$$

The process G also has a multiplicative decomposition given by $G = L^{\mathbb{F}} e^{-\int_0^\cdot \lambda^{\mathbb{F}}_s \eta(ds)}$.

The process $\lambda^{\mathbb{F}}$ is referred to as the $\mathbb{F}$-intensity of τ. The process $\lambda^{\mathbb{G}}$ defined by $\lambda^{\mathbb{G}}_t = \lambda^{\mathbb{F}}_t 1_{\{t\leq\tau\}},\ t \in \mathbb{R}^+$, is the $\mathbb{G}$-intensity of τ. The compensator $\Lambda^{\mathbb{G}}$ of D w.r.t. $\mathbb{G}$ is given by $\Lambda^{\mathbb{G}}_t = \int_0^t \lambda^{\mathbb{G}}_s \eta(ds), t \in \mathbb{R}^+$. Let $N^{\mathbb{G}}$ be the pure jump $\mathbb{G}$-martingale defined by

$$N^{\mathbb{G}} \;=\; D - \Lambda^{\mathbb{G}}. \tag{5.6}$$

Note that τ is a totally inaccessible $\mathbb{G}$-stopping time since the compensator $\Lambda^{\mathbb{G}}$ is continuous at τ (see [Pro04, Theorems 20 and 21 of Chapter 3]). Throughout this chapter, we suppose that

Assumption 5.2.5. *The process $\lambda = \lambda^{\mathbb{F}} a^{\mathbb{F}}$ is essentially bounded on $[0,T] \times \Omega$, i.e.*

$$|\lambda|_\infty := \operatorname*{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega} |\lambda_t(\omega)| < +\infty.$$

As $G \leq 1$ and $L^{\mathbb{F}}_t = G_t e^{\int_0^t \lambda_s ds}, t \in [0,T]$, Assumption 5.2.5 implies that $(L^{\mathbb{F}}_t)_{t\in[0,T]}$ is an $(\mathbb{F},\mathbb{P})$-martingale. Let $\widehat{\mathbb{Q}}$ be the probability measure defined on $(\Omega,\mathcal{A})$ defined by

$$\widehat{\mathbb{Q}}(A) := \mathbb{E}\left[L^{\mathbb{F}}_T 1_A\right], A \in \mathcal{A}. \tag{5.7}$$

Observe that $\widehat{\mathbb{Q}}$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and has Radon-Nikodym density given by $(L^{\mathbb{F}}_t)_{t\in[0,T]}$. The measure $\widehat{\mathbb{Q}}$ will be very useful in Section 5.3 in providing the pre-default value of the first component of the solution to RBSDEs in the filtration $\mathbb{G}$. A standard hypothesis in the theory of progressive enlargement which guarantees the preservation of the semimartingale invariance property is the *immersion* or $(\mathcal{H})$-hypothesis (see [JLC09a, CJN12, Kus99]).

Assumption 5.2.6. *[$(\mathcal{H})$-hypothesis] Every $(\mathbb{F},\mathbb{P})$-martingale is a $(\mathbb{G},\mathbb{P})$-martingale.*

Remark 5.2.7. *Due to the density hypothesis and Assumption 5.2.3, the immersion hypothesis is equivalent to $\alpha^d_t(u) = \alpha^d_u(u), \mathbb{P}$-a.s. for any $t \geq u, u \in \mathbb{R}^+$ (see [JLC09b, Corollary 3.1]). We infer from (5.4) that G is therefore increasing under the immersion hypothesis and and Proposition 5.2.4 implies that $L^{\mathbb{F}} = 1$. Hence $\widehat{\mathbb{Q}} = \mathbb{P}$ if the immersion hypothesis holds.*

We close this section by introducing normed spaces that we will use. For $p \geq 1, n \in \mathbb{N}$, $\mathbb{H} \in \{\mathbb{F}, \mathbb{G}^\tau, \mathbb{G}\}$, $\mathbb{Q}$ a probability measure on $(\Omega,\mathcal{A})$ equivalent to $\mathbb{P}$ and an $\mathbb{R}^n$-valued martingale $M = (M^i)_{1\leq i\leq n}$ an $\mathbb{R}^n$-valued càdlàg $(\mathbb{H},\mathbb{Q})$-local martingale, we consider the following spaces:

- $L^p(\mathcal{H}_T,\mathbb{Q})$ (resp. $L^\infty(\mathcal{H}_T)$) denote the space of $\mathcal{H}_T$-measurable random variables ξ with

$$||\xi||^p_{L^p(\mathcal{H}_T,\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}}\left[|\xi|^p\right] < +\infty \left(\text{resp. } |\xi|_\infty := \operatorname*{ess\,sup}_{\omega\in\Omega} |\xi(\omega)| < +\infty\right).$$

- $\mathcal{S}^p_T(\mathbb{H},\mathbb{Q})$ (resp. $\mathcal{S}^\infty_T(\mathbb{H})$) the space of real valued $\mathbb{H}$-adapted càdlàg processes Y such that

$$||Y||^p_{\mathcal{S}^p_T(\mathbb{H},\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}}\left[\left(\sup_{t\in[0,T]} |Y_t|\right)^p\right] < +\infty \left(\text{resp. } ||Y||_\infty := \left|\sup_{t\in[0,T]} |Y_t|\right|_\infty < +\infty\right).$$

- $\mathcal{IS}_T(\mathbb{H})$ denotes the space of real valued $\mathbb{H}$-predictable increasing processes K with initial value 0, while $\mathcal{IS}^p_T(\mathbb{H},\mathbb{Q}) = \mathcal{S}^p_T(\mathbb{H},\mathbb{Q}) \cap \mathcal{IS}_T(\mathbb{H})$.

- $L_{\mathbb{H}}(M)$ denotes the space of $\mathbb{R}^n$-valued $\mathbb{H}$-predictable processes Z such that the stochastic integral $\int ZdM$ w.r.t. $\mathbb{H}$ is well defined.

- $\mathcal{H}^{p,n}_T(\mathbb{H},\mathbb{Q})$ the space of $\mathbb{R}^n$-valued $\mathcal{P}(\mathbb{H})$- measurable processes Z such that

$$||Z||^p_{\mathcal{H}^{p,n}(\mathbb{H},\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T ||Z_s||^2 ds\right)^{\frac{p}{2}}\right] < \infty.$$

- $\mathcal{L}^p_T(\mathbb{G},\mathbb{Q})$ the space of $\mathbb{G}$-predictable processes ζ such that

$$||\zeta||^p_{\mathcal{L}^p(\mathbb{G},\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T |\zeta_s|^2 dD_s\right)^{\frac{p}{2}}\right] < +\infty.$$

- $\mathcal{M}^{loc}_{\mathbb{H}}(M,\mathbb{Q})$ is the space of real valued $(\mathbb{H},\mathbb{Q})$-local martingales N such that $[N, M^i] = 0$ for $i = 1, 2, \cdots, n$.

- $\mathcal{M}^p_{\mathbb{H}}(M,\mathbb{Q})$ the subspace of $\mathcal{M}^{loc}_{\mathbb{H}}(M,\mathbb{Q})$ consisting of processes N satisfying

$$||N||^p_{\mathcal{M}^p(\mathbb{H},\mathbb{Q})} := \mathbb{E}^{\mathbb{Q}}\left[[N]_T^{\frac{p}{2}}\right] < +\infty.$$

For a real valued $\mathbb{H}$-optional process ζ, $|\zeta|_\infty := \operatorname{ess\,sup}_{(t,\omega)\in[0,T]\times\Omega}|\zeta_t(\omega)|$.

The following lemma complements Lemma 4.2.16 by providing a bridge between the spaces $\mathcal{S}^r_T(\mathbb{F},\mathbb{P})$ (resp. $\mathcal{H}^{r,n}_T(\mathbb{F},\widehat{\mathbb{Q}})$) and $\mathcal{S}^r_T(\mathbb{G},\mathbb{P})$ (resp. $\mathcal{H}^{r,n}_T(\mathbb{G},\mathbb{P})$) under Assumption 5.2.5 and with $\widehat{\mathbb{Q}}$ defined by (5.7).

Lemma 5.2.8. *Let $r\geq 1$ and $D_t = 1_{\{t\geq\tau\}}$, $t\in[0,T]$.*

a) Let Y be a $\mathbb{G}$-adapted càdlàg process such that $Y_t = Y^b_t 1_{\{t<\tau\}} + Y^d_t(\tau)1_{\{t\geq\tau\}}$, $t\in[0,T]$, where Y^b is an $\mathcal{O}(\mathbb{F})$-measurable process and Y^d an $\mathcal{O}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable process.

i) If $Y\in\mathcal{S}^r_T(\mathbb{G},\mathbb{P})$, then $Y^b\in\mathcal{S}^r_T(\mathbb{F},\widehat{\mathbb{Q}})$ and $Y^d(\tau)D\in\mathcal{S}^r_T(\mathbb{G}^\tau,\mathbb{P})$.

ii) If $Y^b\in\mathcal{S}^r_T(\mathbb{F},\widehat{\mathbb{Q}})$, then $Y^b(1-D)\in\mathcal{S}^r_T(\mathbb{G},\mathbb{P})$ and $Y^b_\tau 1_{\{T\geq\tau\}}\in L^r(\mathcal{G}_T,\mathbb{P})$.

iii) If $r\geq 2$, $Y^d(\tau)\in\mathcal{S}^r_T(\mathbb{G}^\tau,\mathbb{P})$ and $Y^d_\cdot(\cdot)$ is predictable[1] , then we have $\sqrt{\lambda}Y^d_\cdot(\cdot)\in\mathcal{H}^{r,1}_T(\mathbb{F},\widehat{\mathbb{Q}})$.

b) Let Z be an $\mathbb{R}^n$-valued $\mathbb{G}$-predictable process such that $Z_t = Z^b_t 1_{\{t\leq\tau\}} + Z^d_t(\tau)1_{\{t>\tau\}}$, $t\in[0,T]$, where Z^b (resp. Z^d) is a $\mathcal{P}(\mathbb{F})$ (resp. $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$)-measurable process.

i) If $Z\in\mathcal{H}^{r,n}_T(\mathbb{G},\mathbb{P})$ then $Z^b\in\mathcal{H}^{r,n}_T(\mathbb{F},\widehat{\mathbb{Q}})$ and $Z^d(\tau)D_-\in\mathcal{H}^{r,n}_T(\mathbb{G}^\tau,\mathbb{P})$.

ii) $Z^b\in\mathcal{H}^{r,n}_T(\mathbb{F},\widehat{\mathbb{Q}})$ if and only if $Z^b(1-D_-)\in\mathcal{H}^{r,n}_T(\mathbb{G},\mathbb{P})$.

c) Let ξ^b be an $\mathcal{F}_T$-measurable random variable. Then $\xi^b 1_{\{T<\tau\}}\in L^r(\mathcal{G}_T,\mathbb{P})$ if and only $\xi^b\in L^r(\mathbb{F},\widehat{\mathbb{Q}})$.

Proof. Let $C = |\lambda|_\infty$. We recall that for $t\in[0,T]$

$$G_t = \mathbb{P}(\tau>t|\mathcal{F}_t)\leq 1,\quad L^{\mathbb{F}}_t = G_t e^{\int_0^t\lambda_u du}\ \text{ and }\ \alpha^{\mathrm{d}}_t(t)\eta(dt) = \lambda_t G_t dt.$$

a). We start with i). For $t\in[0,T]$, $|Y_t| = |Y^{\mathrm{b}}_t(1-D_t)| + |Y^{\mathrm{d}}_t(\tau)D_t|$. Hence $Y^{\mathrm{d}}(\tau)D\in\mathcal{S}^r_T(\mathbb{G}^\tau,\mathbb{P})$. To show that $Y^{\mathrm{b}}\in\mathcal{S}^r_T(\mathbb{F},\widehat{\mathbb{Q}})$, we consider the $\mathbb{F}$-adapted increasing process ζ defined for $t\in[0,T]$ by

$$\zeta_t = \sup_{s\leq t}|Y^{\mathrm{b}}_s|^r. \tag{5.8}$$

Note that $L^{\mathbb{F}}_t\leq e^{CT}G_t, t\in[0,T]$, and $\zeta_T 1_{\{T<\tau\}}\leq\sup_{t\in[0,T]}|Y^{\mathrm{b}}_t 1_{\{t<\tau\}}|^r$. Hence, applying Bayes' formula we obtain that

$$\begin{aligned}\mathbb{E}^{\widehat{\mathbb{Q}}}[\zeta_T] &= \mathbb{E}\left[L^{\mathbb{F}}_T\zeta_T\right]\leq e^{CT}\mathbb{E}[G_T\zeta_T] = e^{CT}\mathbb{E}\left[\zeta_T 1_{\{T<\tau\}}\right]\leq e^{CT}\mathbb{E}\left[\sup_{t\in[0,T]}|Y^{\mathrm{b}}_t 1_{\{t<\tau\}}|^r\right]\\ &\leq e^{CT}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^r\right] < +\infty.\end{aligned}$$

We therefore have $Y^{\mathrm{b}}\in\mathcal{S}^r_T(\mathbb{F},\widehat{\mathbb{Q}})$.

ii). Let ζ be given by (5.8). By hypothesis, $\mathbb{E}^{\widehat{\mathbb{Q}}}[\zeta_T] < +\infty$. Observe that for $t\in[0,T]$, one has

$$|Y^{\mathrm{b}}_t(1-D_t)|^r = |Y^{\mathrm{b}}_t 1_{\{t<\tau\}}|^r \leq |Y^{\mathrm{b}}_{t\wedge\tau}|^r\leq\zeta_{t\wedge\tau}\leq\zeta_{T\wedge\tau}.$$

[1] The process $Y^{\mathrm{d}}_\cdot(\cdot)$ is optional and the space $\mathcal{H}^{r,1}_T(\mathbb{F},\widehat{\mathbb{Q}})$ is defined only for predictable processes.

Using the above inequality, the density hypothesis, the inequality $\alpha_t^{\mathrm{d}}(t) = \lambda_t^{\mathbb{F}} G_t \le C L_t^{\mathbb{F}}, t \in [0,T]$, and Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t^{\mathrm{b}} 1_{\{t<\tau\}}|^r\right] &\le \mathbb{E}\left[\zeta_T 1_{\{T<\tau\}} + \zeta_\tau 1_{\{T\ge\tau\}}\right] = \mathbb{E}\left[\zeta_T G_T + \int_0^T \zeta_u \alpha_u^{\mathrm{d}}(u)\eta(du)\right] \\ &\le \mathbb{E}\left[\zeta_T L_T^{\mathbb{F}} + C\int_0^T \zeta_u L_u^{\mathbb{F}} du\right] \le \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\zeta_T\right] + C\int_0^T \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\zeta_u\right] du \\ &\le \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\zeta_T\right](1+CT) < +\infty. \end{aligned}$$

The last inequality ensures that $Y^{\mathrm{b}}(1-D) \in \mathcal{S}_T^r(\mathbb{G},\mathbb{P})$. Using similar arguments, one shows that $Y_\tau^{\mathrm{b}} 1_{\{T\ge\tau\}} \in L^r(\mathcal{G}_T,\mathbb{P})$.

iii) Now assume that $r \ge 2$ and $Y^{\mathrm{d}}(\tau) \in \mathcal{S}_T^r(\mathbb{G}^\tau,\mathbb{P})$. Then $\mathbb{E}\left[|Y_\tau^{\mathrm{d}}(\tau)|^r 1_{\{T\ge\tau\}}\right] < +\infty$. Applying the density hypothesis, Fubini's theorem and Bayes' formula, we get

$$\begin{aligned} \mathbb{E}\left[|Y_\tau^{\mathrm{d}}(\tau)|^r 1_{\{T\ge\tau\}}\right] &= \mathbb{E}\left[\int_0^T |Y_u^{\mathrm{d}}(u)|^r \lambda_u L_u^{\mathbb{F}} e^{-\int_0^u \lambda_s ds} du\right] \\ &= \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\int_0^T |Y_u^{\mathrm{d}}(u)|^r \lambda_u e^{-\int_0^u \lambda_s ds} du\right] < +\infty. \end{aligned}$$

As $|\lambda|_\infty < C$, the above inequality implies that $\mathbb{E}^{\widehat{\mathbb{Q}}}\left[\int_0^T |Y_u^{\mathrm{d}}(u)|^r \lambda_u du\right] < +\infty$. For $r = 2$, the latter inequality is equivalent to $\sqrt{\lambda} Y_\cdot^{\mathrm{d}}(\cdot) \in \mathcal{H}_T^{2,n}(\mathbb{F},\widehat{\mathbb{Q}})$. For $r > 2$ we have $\frac{r}{2} > 1$. Again the latter inequality and Hölder's inequality lead to

$$\begin{aligned} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\int_0^T \lambda_s |Y_u^{\mathrm{d}}(u)|^2 ds\right)^{\frac{r}{2}}\right] &\le \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\int_0^T \lambda_u^{\frac{r}{2}} |Y_u^{\mathrm{d}}(u)|^r du\right)\right] \\ &\le C^{\frac{r}{2}-1}\mathbb{E}\left[\int_0^T \lambda_u |Y_u^{\mathrm{d}}(u)|^r du\right) < +\infty. \end{aligned}$$

We deduce that $\sqrt{\lambda} Y_\cdot^{\mathrm{d}}(\cdot) \in \mathcal{H}_T^{r,n}(\widehat{\mathbb{F}},\mathbb{Q})$.

b) We start with i). As $|Z_t| = |Z_t^{\mathrm{b}}(1-D_{t-})| + |Z_t^{\mathrm{d}}(\tau)D_{t-}|,\ t \in [0,T]$, we have $Z^{\mathrm{d}}(\tau)D_- \in \mathcal{H}_T^{r,n}(\mathbb{G}^\tau,\mathbb{P})$ and $Z^{\mathrm{b}}(1-D_-) \in \mathcal{H}_T^{r,n}(\mathbb{G},\mathbb{P})$ if $Z \in \mathcal{H}_T^{r,n}(\mathbb{G},\mathbb{P})$. Let us now show that $Z^{\mathrm{b}} \in \mathcal{H}_T^{r,n}(\mathbb{F},\widehat{\mathbb{Q}})$. Applying the density hypothesis, we obtain

$$\begin{aligned} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\left(\int_0^T ||Z_s^{\mathrm{b}}||^2 ds\right)^{\frac{r}{2}}\right] &= \mathbb{E}\left[\left(\int_0^T ||Z_s^{\mathrm{b}}||^2 ds\right)^{\frac{r}{2}} L_T^{\mathbb{F}}\right] \le \mathbb{E}\left[\left(\int_0^T ||Z_s^{\mathrm{b}}||^2 ds\right)^{\frac{r}{2}} 1_{\{T<\tau\}}\right] \\ &\le \mathbb{E}\left[\left(\int_0^{T\wedge\tau} ||Z_s^{\mathrm{b}}||^2 1_{\{s\le\tau\}} ds\right)^{\frac{r}{2}} 1_{\{T<\tau\}}\right] \\ &\le \mathbb{E}\left[\left(\int_0^T ||Z_s^{\mathrm{b}}(1-D_{s-})||^2 ds\right)^{\frac{r}{2}}\right]. \end{aligned}$$

We deduce that $Z^{\mathrm{b}} \in \mathcal{H}_T^{r,n}(\mathbb{F},\widehat{\mathbb{Q}})$ if $Z \in \mathcal{H}_T^{r,n}(\mathbb{G},\mathbb{P})$.

Next we show ii). Assuming that $Z^{\mathrm{b}} \in \mathcal{H}_T^{r,n}(\mathbb{F},\widehat{\mathbb{Q}})$, an application of the density hypothesis, Fubini's theorem, Bayes's formula and the inequalities $G_T \le L_T^{\mathbb{F}},\ \alpha_s^{\mathrm{d}}(s)\eta(ds) \le CL_s^{\mathbb{F}} ds,\ s \in [0,T]$

yield

$$\begin{aligned}||Z^{\mathrm{b}}(1-D_-)||^r_{\mathcal{H}^{r,n}_T(\mathbb{G},\mathbb{P})} &= \mathbb{E}\left[\left(\int_0^T ||Z^{\mathrm{b}}_s||^2 1_{\{s\le\tau\}}ds\right)^{\frac{r}{2}} 1_{\{T<\tau\}}\right] + \mathbb{E}\left[\left(\int_0^{T\wedge\tau} ||Z^{\mathrm{b}}_s||^2 ds\right)^{\frac{r}{2}} 1_{\{T\ge\tau\}}\right] \\ &= \mathbb{E}\left[\left(\int_0^T ||Z^{\mathrm{b}}_s||^2 ds\right)^{\frac{r}{2}} G_T + \left(\int_0^\tau ||Z^{\mathrm{b}}_s||^2 ds\right)^{\frac{r}{2}} 1_{\{T\ge\tau\}}\right] \\ &\le \mathbb{E}\left[\left(\int_0^T ||Z^{\mathrm{b}}_s||^2 ds\right)^{\frac{r}{2}} L^{\mathbb{F}}_T\right] + \mathbb{E}\left[\int_0^T \left(\int_0^s ||Z^{\mathrm{b}}_u||^2 du\right)^{\frac{r}{2}} \alpha^{\mathrm{d}}_s(s)\eta(ds)\right] \\ &\le ||Z^{\mathrm{b}}||^r_{\mathcal{H}^{r,n}_T(\mathbb{F},\widehat{\mathbb{Q}})}(1+CT) < +\infty.\end{aligned}$$

c) This follows from similar arguments as in a) and b). □

For the introduction of further normed spaces, let us recall the definition of BMO martingales.

Definition 5.2.9. *Let $\mathbb{H} \in \{\mathbb{F},\mathbb{G},\mathbb{G}^\tau\}$ and $\mathbb{Q}$ be a probability measure on $(\Omega,\mathcal{A})$. An $(\mathbb{H},\mathbb{Q})$-local martingale N with $N_0 = 0$ is a BMO martingale if there exists a constant $C > 0$, such that for every $\nu \in \mathcal{T}$, we have*

$$\mathbb{E}^{\mathbb{Q}}\left[[N]_T - [N]_{\nu_-}|\mathcal{H}_\nu\right] \le C^2 \ \mathbb{Q}\text{-}a.s.$$

The smallest constant C for which the above inequality is satisfied is the BMO-norm of N. We denote it by $||N||_{BMO(\mathbb{H},\mathbb{Q})}$ or simply $||N||_{BMO}$ if there is no ambiguity. Regarding properties of BMO martingales and equivalent definitions, we refer to [Kaz94, DDM79, ISS79]. The following spaces of processes are linked to BMO martingales:

- $\mathcal{H}^{2,n}_{BMO}(\mathbb{H},\mathbb{Q}) = \left\{Z \in \mathcal{H}^{2,n}_T(\mathbb{H},\mathbb{Q}) \text{ with } \sup_{t\in[0,T]}\left\|\mathbb{E}^{\mathbb{Q}}\left[\int_t^T ||Z_s||^2 ds|\mathcal{H}_t\right]\right\|_\infty < +\infty\right\}$,
- $\mathcal{L}^2_{BMO}(\mathbb{G},\mathbb{Q}) = \left\{\zeta \in \mathcal{L}^2(\mathbb{G},\mathbb{Q}) \text{ such that } ||\zeta\cdot N^{\mathbb{G}}||_{BMO} < +\infty\right\}$,
- $\mathcal{IS}_{BMO}(\mathbb{H},\mathbb{Q}) = \left\{X \in \mathcal{IS}_T(\mathbb{H}) \text{ with } \sup_{t\in[0,T]}||\mathbb{E}^{\mathbb{Q}}\left[X_T - X_t|\mathcal{H}_t\right]||_\infty < +\infty\right\}$.

5.2.2 RBSDEs in a progressively enlarged filtration

In this section, we introduce RBSDEs in the filtration $\mathbb{G}$ which we investigate in this chapter. In order to enrich the structure of the class we consider, we make the following standing assumption on $\mathbb{F}$:

Assumption 5.2.10. *The filtration $\mathbb{F}$ supports an $\mathbb{R}^n$-valued Brownian motion $B = (B^i)_{1\le i\le n}$ on $[0,T]$.*

Assumption 5.2.1 ensures that B is a $\mathbb{R}^n$-valued $\mathbb{G}$-semimartingale. By Theorem 3.1 in [JLC09b], its $(\mathbb{G},\mathbb{P})$-local martingale part $B^{\mathbb{G}}$ is given by

$$B^{\mathbb{G}}_t = B_t - \int_0^{t\wedge\tau} \frac{d\langle B,G\rangle_s}{G_{s-}} - \int_{t\wedge\tau}^t \frac{d\langle B,\alpha^{\mathrm{d}}(\tau)\rangle_s}{\alpha^{\mathrm{d}}_{s-}(\tau)},\ t\in[0,T]. \tag{5.9}$$

Note that $B^{\mathbb{G}}$ has continuous paths and for $t \in [0,T]$ we have: $[B^{\mathbb{G},i}, B^{\mathbb{G},j}]_t = [B^i, B^j]_t = t$ if $i = j$ and 0 if not. Consequently, $B^{\mathbb{G}}$ is a $(\mathbb{P},\mathbb{G})$-Brownian motion.

We now introduce the following objects that will play the role of input data in the sequel.

a) A terminal value $\xi \in L^1(\mathcal{G}_T, \mathbb{P})$ which is of the form

$$\xi = \xi^{\mathrm{b}} 1_{\{T<\tau\}} + \xi^{\mathrm{d}}(\tau) 1_{\{T \geq \tau\}} \in L^1(\mathcal{G}_T, \mathbb{P}), \tag{5.10}$$

where ξ^{b} is an $\mathcal{F}_T$-measurable random variable and $\xi^{\mathrm{d}}(\tau)$ a $\mathcal{G}_T^{\tau}$-measurable random variable.

b) An obstacle process $S \in \mathcal{S}_T^1(\mathbb{G}, \mathbb{P})$ satisfying $S_T \leq \xi$ and with optional splitting formula

$$S_t = S_t^{\mathrm{b}} 1_{\{t<\tau\}} + S_t^{\mathrm{d}}(\tau) 1_{\{t \geq \tau\}}, \; t \in [0, T, \tag{5.11}$$

where S^{b} is an $\mathcal{O}(\mathbb{F})$-measurable process and S^{d} and $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process.

c) A driver $F : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ which is $\mathcal{P}(\mathbb{G}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}) - \mathcal{B}(\mathbb{R})$-measurable. There exists $F^{\mathrm{b}} : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n) \otimes \mathcal{B}(\mathbb{R}) - \mathcal{B}(\mathbb{R})$-measurable and $F^{\mathrm{d}} : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \to \mathbb{R}$ $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n) - \mathcal{B}(\mathbb{R})$-measurable such that for $(\omega, t, y, z, u) \in [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}$,

$$F(t, \omega, y, z, u) \quad = \quad F^{\mathrm{b}}(t, \omega, y, z, u) 1_{\{t \leq \tau(\omega)\}} + F^{\mathrm{d}}(t, \omega, y, z) 1_{\{t > \tau(\omega)\}}. \tag{5.12}$$

Definition 5.2.11. *Let $N^{\mathbb{G}}$ be the pure jump martingale defined by (5.6). We say that quintuplet of processes (Y, Z, U, M, K) is a solution to the RBSDE with driver F, obstacle process S and terminal value ξ (hereafter denoted by RBSDE(F, S, ξ)) if*

i) Y is real valued càdlàg $\mathbb{G}$-semimartingale, $Z \in L_{\mathbb{G}}(B^{\mathbb{G}})$ and $U \in L_{\mathbb{G}}(N^{\mathbb{G}})$,

ii) K is a $\mathbb{G}$-predictable increasing process with càdlàg paths,

iii) M is a $(\mathbb{G}, \mathbb{P})$-local martingale belonging to the space $\mathcal{M}_{\mathbb{G}}^{loc}(B^{\mathbb{G}}, \mathbb{P}) \cap \mathcal{M}_{\mathbb{G}}^{loc}(N^{\mathbb{G}}, \mathbb{P})$,

iv) $\int_0^T |F(s, Y_{s-}, Z_s, U_s)| ds$ is finite $\mathbb{P}$-a.s.

v) (Y, Z, U, M, K) satisfies for $t \in [0, T]$

$$\begin{cases} Y_t = \xi + \int_t^T F(s, Y_{s-}, Z_s, U_s) ds + \int_t^T dK_s - \int_t^T Z_s dB_s^{\mathbb{G}} - \int_t^T U_s dN_s^{\mathbb{G}} - \int_t^T dM_s, \\ Y_t \geq S_t, \\ \int_0^T (Y_{t-} - S_{t-}) dK_t = 0. \end{cases} \tag{5.13}$$

We will also refer to the equation (5.13) as RBSDE(F, S, ξ) and the triplet (F, S, ξ) as the data of (5.13).

Remark 5.2.12. *The condition $\int_0^T (Y_{t-} - S_{t-}) dK_t = 0$ is known in the literature as the Skorohod condition. When Y and S are continuous, it implies that K is non constant only when Y hits S and K acts as a* push *to keep Y above S and this in a minimal way.*

For $p > 1$, we introduce the following two additional spaces:

- $\mathcal{M}_T^p(\mathbb{G}, \mathbb{P}) = \mathcal{M}_{\mathbb{G}}^p(B^{\mathbb{G}}, \mathbb{P}) \cap \mathcal{M}_{\mathbb{G}}(N^{\mathbb{G}}, \mathbb{P})$,
- $\mathcal{S}_{\mathbf{sol}}^p(\mathbb{G}, \mathbb{P}) = \mathcal{S}_T^p(\mathbb{G}, \mathbb{P}) \times \mathcal{H}_T^{p,n}(\mathbb{G}, \mathbb{P}) \times \mathcal{L}_T^p(\mathbb{G}, \mathbb{P}) \times \mathcal{M}_T^p(\mathbb{G}, \mathbb{P}) \times \mathcal{IS}_T^p(\mathbb{G}, \mathbb{P})$.

Definition 5.2.13. *Let $p > 1$. We say that a solution (Y, Z, U, M, K) to (5.13) is*

- *an $\mathcal{S}^p$-solution if $(Y, Z, U, M, K) \in \mathcal{S}_{\mathbf{sol}}^p(\mathbb{G}, \mathbb{P})$,*
- *a bounded solution if $Y \in \mathcal{S}_T^{\infty}(\mathbb{G})$.*

Let us recall the following well known representation of an $\mathcal{S}^p$-solution of (5.13) which connects RBSDEs and the notion of Snell envelope.

Proposition 5.2.14. *Let $p > 1$ and (Y, Z, U, M, K) be an $\mathcal{S}^p$-solution to (5.13). Then for every $\nu \in \mathcal{T}_T(\mathbb{G})$,*

$$Y_\nu \quad = \quad \operatorname*{ess\,sup}_{\sigma \in \mathcal{T}_{\nu,T}(\mathbb{G})} \mathbb{E}\left[\xi 1_{\{\sigma=T\}} + S_\sigma 1_{\{\sigma<T\}} + \int_\sigma^T F(s, Y_s, Z_s, U_s) \Big| \mathcal{G}_\nu\right]. \tag{5.14}$$

Proof. See [LX05, Proposition 3.1]. □

From (5.14), given an $\mathcal{S}^p$-solution (Y, Z, U, M, K), the process $Y + \int_0^\cdot F(s, Y_{s-}, Z_s, U_s)ds$ is the Snell envelope of $\zeta + \int_0^\cdot F(s, Y_{s-}, Z_s, U_s)ds$ where $\zeta_t = S_t 1_{\{t<T\}} + \xi 1_{\{t=T\}}, t \in [0, T]$.

5.2.3 Problem formulation and motivation

Given a solution (Y, Z, U, M, K) of (5.13) when it exists, Proposition 5.2.2 postulates an optional splitting of the solution, i.e. the existence of a quintuplet $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, U^{\mathrm{b}}, M^{\mathrm{b}}, K^{\mathrm{b}})$ of $\mathcal{O}(\mathbb{F})$-measurable processes and a quintuplet $(Y^{\mathrm{d}}, Z^{\mathrm{d}}, U^{\mathrm{d}}, M^{\mathrm{d}}, K^{\mathrm{d}})$ of $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable processes such that for $t \in [0, T]$

$$\begin{cases} Y_t &= Y_t^{\mathrm{b}} 1_{\{t<\tau\}} + Y_t^{\mathrm{d}}(\tau) 1_{\{t\geq\tau\}}, \\ Z_t &= Z_t^{\mathrm{b}} 1_{\{t\leq\tau\}} + Z_t^{\mathrm{d}}(\tau) 1_{\{t>\tau\}}, \\ U_t &= U_t^{\mathrm{b}} 1_{\{t\leq\tau\}} + U_t^{\mathrm{d}}(\tau) 1_{\{t>\tau\}}, \\ M_t &= M_t^{\mathrm{b}} 1_{\{t<\tau\}} + M_t^{\mathrm{d}}(\tau) 1_{\{t\geq\tau\}}, \\ K_t &= K_t^{\mathrm{b}} 1_{\{t\leq\tau\}} + K_t^{\mathrm{d}}(\tau) 1_{\{t>\tau\}}. \end{cases}$$

Our primary goal in this chapter is to give a description of the quintuplets $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, U^{\mathrm{b}}, M^{\mathrm{b}}, K^{\mathrm{b}})$ and $(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), U^{\mathrm{d}}(\tau), M^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau))$ which form the pre-default and post-default values of the solution (Y, Z, U, M, K). This description will lead to a methodology to construct solutions to (5.13) which consists in looking for $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, M^{\mathrm{b}}, K^{\mathrm{b}})$ as a solution to a suitable RBSDE in the filtration $\mathbb{F}$, $(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), M^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau))$ as a solution to a RBSDE in the filtration $\mathbb{G}^\tau$ and obtain (Y, Z, U, M, K) by suitably pasting the solutions $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, M^{\mathrm{b}}, K^{\mathrm{b}})$ and $(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), M^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau))$ at time τ (see Theorem 5.3.5 for a precise formulation). The methodology explained is rather intuitive as we expect the pre-default and post-default values of (Y, Z, U, M, K) to be of the same nature, i.e. solutions to RBSDEs. Moreover, $\mathbb{G}$ can be identified with $\mathbb{F}$ and $\mathbb{G}^\tau$ respectively before and after τ. The delicate task is to identify the respective RBSDEs for the pre-default and post-default values of the solution (Y, Z, U, M, K) and to justify the pasting procedure as we deal with different filtrations and stochastic integration is not stable under a change of filtration [Jeu80, CMS80, Jac80, Wei84]. We will rely on the link between RBSDEs and the Snell envelope representation of their solution given by Proposition 5.2.14 and the optional splitting formula of the Snell envelope obtained in the previous chapter to identify the appropriate set of data. Regarding the pasting procedure, we will make use of the integration formula established in Lemma 5.3.8 below.

Let us point out that the existing results for general filtrations [Kli15, BPTZ15] or filtrations containing jumps [QS14, HO16, Ess08] which yield the existence of an $\mathcal{S}^p$-solution to the RBSDE (5.13) do not give an optional splitting formula of the solution. Striving to obtain a solution to (5.13) by identifying its pre-default and post-defaults values allows to circumvent the additional difficulty posed by the pure jump martingale $N^{\mathbb{G}}$ when addressing the existence result directly in the filtration $\mathbb{G}$. Indeed, as one might expect, the RBSDEs for the pre-default and post-default values are not driven anymore by the jump martingale $N^{\mathbb{G}}$ and are easier to solve as they contain one less jump. As a result, we will obtain existence results for a larger class of drivers than those treated in the current literature [Kli15, BPTZ15, Lio14, KLQT02], see Theorem 5.3.13 for F Lipschitz and Theorem 5.4.24 for F having quadratic growth in z and exponential growth in u. From a numerical perspective, the construction procedure by pasting is advantageous as there exist very few numerical schemes for RBSDEs with jumps [DL16a, DL16b]. Indeed, if the filtrations $\mathbb{F}$ and $\mathbb{G}^\tau$ are continuous [2], then the resulting RBSDEs for both the pre-default and post-default values are not driven by jump martingales and one can therefore apply the numerical scheme for RBSDEs without jumps [MPX08, Xu11, Cha09] to obtain a numerical approximation of (Y, Z, U, M, K). In practice, solutions to RBSDEs are connected to the value functions, optimal trading strategies and/or optimal stopping times to stochastic control problems such as optimal stopping problems [EKKP+97, LX05], risk sensitive mixed control problems [Ham02, EKH03], pricing and hedging of American options [EKPQ97a, LMX05, KLQT02], mixed game problem [HL00], controller-stopper problems [BZ14]. The precise knowledge of the pre-default and the post-default values of (Y, Z, U, M, K) will lead to a suitable understanding of the optimal policies both before and after default in the aforementioned problems. This has already been illustrated in the particular case of the optimal stopping problem where the knowledge of the pre-default and post-default values of the Snell envelope is necessary for the characterization of optimal stopping times, see Theorem 4.3.7. See also Theorem 4.4.13 for the importance of the pre-default and post-default values of the Snell envelope for the construction of hedging strategies for American options.

5.3 Decomposition approach for solving RBSDEs in $\mathbb{G}$

Our focus in this section is to develop a decomposition approach to construct a solution to the RBSDE(F, S, ξ). This approach consists in solving alternative RBSDEs in the filtrations $\mathbb{F}$ and $\mathbb{G}^\tau$ whose solutions correspond respectively to the pre-default and post-default values of the solution to the RBSDE(F, S, ξ). Before motivating the choices of the alternative RBSDEs, let us introduce two additional processes $B^{\mathbb{F}}$ and $B^{\mathbb{G}^\tau}$ that will play the role of underlying martingales in the filtrations $\mathbb{F}$ and $\mathbb{G}^\tau$ respectively. We recall that B is an $\mathbb{R}^n$-valued $\mathbb{F}$-Brownian motion and $L^{\mathbb{F}}$ is given by (5.5). Let

$$B^{\mathbb{F}}_t = B_t - \int_0^t \frac{d\langle B, L^{\mathbb{F}}\rangle_s}{L^{\mathbb{F}}_{s-}}, \ t \in [0, T], \tag{5.15}$$

$$B^{\mathbb{G}^\tau}_t = B_t - \int_0^t \frac{d\langle B, \alpha^{\mathrm{d}}(u)\rangle_s}{\alpha^{\mathrm{d}}_{s-}(u)}\bigg|_{u=\tau}, \ t \in [0, T]. \tag{5.16}$$

Remark 5.3.1. *Some useful properties of $B^{\mathbb{F}}$ and $B^{\mathbb{G}^\tau}$ have to be pointed out.*

i) By Girsanov's theorem, $B^{\mathbb{F}}$ is an $(\mathbb{F}, \widehat{\mathbb{Q}})$-Brownian motion where $\widehat{\mathbb{Q}}$ is measure defined by (5.7). We recall that under the immersion hypothesis, $L^{\mathbb{F}} = 1$ and thus $B^{\mathbb{F}} = B$.

ii) As $G = e^{-\int_0^\cdot \lambda_s ds} L^{\mathbb{F}}$, it follows that $[B, G] = \int_0^\cdot e^{-\int_0^u \lambda_s ds} d\left[B, L^{\mathbb{F}}\right]_u$ and therefore $\langle B, G\rangle = \int_0^\cdot e^{-\int_0^u \lambda_s ds} d\left\langle B, L^{\mathbb{F}}\right\rangle_u$. Hence $\int_0^\cdot \frac{d\langle B,G\rangle_s}{G_{s-}} = \int_0^\cdot \frac{d\langle B,L^{\mathbb{F}}\rangle_s}{L^{\mathbb{F}}_{s-}}$.

iii) The process $B^{\mathbb{G}^\tau}$ is a $(\mathbb{G}^\tau, \mathbb{P})$-local martingale by [Jac85] or [CJZ13]. Moreover, $B^{\mathbb{G}^\tau}$ is continuous and $\langle B^{\mathbb{G}^\tau}\rangle = \langle B\rangle$. Hence it is a $(\mathbb{G}^\tau, \mathbb{P})$-Brownian motion.

We will also need two additional spaces. For $p \geq 1$ and $\widehat{\mathbb{Q}}$ given by (5.7), let $\mathcal{S}^p_{\mathbf{sol}}(\mathbb{G}^\tau, \mathbb{P})$ and $\mathcal{S}^p_{\mathbf{sol}}(\mathbb{F}, \widehat{\mathbb{Q}})$ be defined as follows:

[2] This is the case if for example $\mathbb{F}$ is the completion of the filtration generated by the Brownian motion B

- $\mathcal{S}^p_{\mathbf{sol}}(\mathbb{G}^\tau,\mathbb{P}) = \mathcal{S}^p_T(\mathbb{G}^\tau,\mathbb{P}) \times \mathcal{H}^{p,n}_T(\mathbb{G}^\tau,\mathbb{P}) \times \mathcal{M}^p_{\mathbb{G}^\tau}(B^{\mathbb{G}^\tau},\mathbb{P}) \times \mathcal{IS}^p_T(\mathbb{G}^\tau,\mathbb{P})$.
- $\mathcal{S}^p_{\mathbf{sol}}(\mathbb{F},\widehat{\mathbb{Q}}) = \mathcal{S}^p_T(\mathbb{F},\widehat{\mathbb{Q}}) \times \mathcal{H}^{p,n}_T(\mathbb{F},\widehat{\mathbb{Q}}) \times \mathcal{M}^p_{\mathbb{F}}(B^{\mathbb{F}},\widehat{\mathbb{Q}}) \times \mathcal{IS}^p_T(\mathbb{F},\widehat{\mathbb{Q}})$.

Let us now formally introduce the two systems of RBSDEs on which we built our algorithm.

Definition 5.3.2. *A quadruplet of processes $\Big(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau)\Big) \in \mathcal{S}^1_{\mathbf{sol}}(\mathbb{G}^\tau,\mathbb{P})$ is said to be a solution to the RBSDE in the filtration $\mathbb{G}^\tau$ associated to the driver F^dD_-, obstacle process $S^d(\tau)D$ and terminal value $\xi^d(\tau)D$ (hereby denoted by RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}$ $(F^dD_-, S^d(\tau)D, \xi^d(\tau)D)$) if*

i) $\int_0^T |F^d(s, Y^d_{s-}(\tau), Z^d_s(\tau))D_{s-}|ds$ *is finite $\mathbb{P}$-a.s.*

ii) $\Big(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau)\Big)$ *satisfies for every $t \in [0,T]$*

$$\begin{cases} dY^d_t(\tau) = -F^d(t, Y^d_{t-}(\tau), Z^d_t(\tau))D_{t-}dt - dK_t(\tau) + Z^d_t(\tau)dB^{\mathbb{G}^\tau}_t - dM^d_t(\tau), \\ Y^d_t(\tau) \geq S^d_t(\tau), \\ Y^d_T(\tau) = \xi^d(\tau)D_T \text{ and } \int_0^T (Y^d_{s-}(\tau) - S^d_{s-}(\tau)D_{s-})dK^d_s(\tau) = 0. \end{cases} \tag{5.17}$$

Let $\Big(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau)\Big)$ be a solution to RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^dD_-, S^d(\tau)D, \xi^d(\tau)D)$. For $p > 1$, we say that $\Big(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau)\Big)$ is an $\mathcal{S}^p$-solution if it belongs to $\mathcal{S}^p_{\mathbf{sol}}(\mathbb{G}^\tau,\mathbb{P})$. We say that $\Big(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau)\Big)$ is a bounded solution if $Y^d(\tau) \in \mathcal{S}^\infty_T(\mathbb{G}^\tau)$.

We will also refer to the system (5.17) as the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^{\rm d}D_-, S^{\rm d}(\tau)D, \xi^{\rm d}(\tau)D_T)$ and to the triplet $(F^{\rm d}D_-, S^{\rm d}(\tau)D, \xi^{\rm d}(\tau)D_T)$ as its input data. We will identify the solution to the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^{\rm d}D_-, S^{\rm d}(\tau)D_-, \xi^{\rm d}(\tau)D_T)$ as the post-default values of a solution to the RBSDE(F, S, ξ).

To define the driver of the RBSDE whose solution yields the pre-default value of a solution to the RBSDE(F, S, ξ), for a real valued $\mathbb{F}$-optional process Γ, we consider the adjusted driver $F^{\rm b}_\Gamma$ defined by

$$F^{\rm b}_\Gamma(t,\omega,y,z) = \quad F^{\rm b}(t,\omega,y,z,\Gamma_t(\omega)-y) + \lambda_t(\Gamma_t(\omega)-y), \;\; (t,\omega,y,z) \in [0,T]\times\Omega\times\mathbb{R}\times\mathbb{R}^n. \tag{5.18}$$

Definition 5.3.3. *Let Γ be a real valued $\mathbb{F}$-optional process and F^b_Γ be defined by (5.18). A quadruplet $(Y^b, Z^b, M^b, K^b) \in \mathcal{S}^1_{\mathbf{sol}}(\mathbb{F},\widehat{\mathbb{Q}})$ is said to be a solution to the RBSDE in the filtration $\mathbb{F}$ and w.r.t. the measure $\widehat{\mathbb{Q}}$ associated to the driver F^b_Γ, obstacle process S^b and terminal value ξ^b (hereafter denoted by RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^b_\Gamma, S^b, \xi^b)$) if*

a) $\int_0^T |F^b_\Gamma(t, Y^b_-, Z^b_t)|dt$ *is finite $\mathbb{P}$-a.s.,*

b) $\Big(Y^b, Z^b, M^b, K^b\Big)$ *satisfies for $t \in [0,T]$*

$$\begin{cases} dY^b_t = -F^b_\Gamma(t, Y^b_{t-}, Z^b_t)dt - dK^b_t + Z^b_t dB^{\mathbb{F}}_t + dM^b_t, \\ Y^b_t \geq S^b_t, \\ Y^b_T = S^b_T \text{ and } \int_0^T (Y^b_{s-} - S^b_{s-})dK^b_s = 0. \end{cases} \tag{5.19}$$

Let (Y^b, Z^b, M^b, K^b) be a solution to RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^b_\Gamma, S^b, \xi^b)$.
For $p \geq 1$, we say that (Y^b, Z^b, M^b, K^b) is an $\mathcal{S}^p$-solution if $(Y^b, Z^b, M^b, K^b) \in \mathcal{S}^p_{\mathbf{sol}}(\mathbb{F},\widehat{\mathbb{Q}})$. The solution (Y^b, Z^b, M^b, K^b) is said to be bounded if $Y^b \in \mathcal{S}^\infty_T(\mathbb{F})$.

We will also refer to the system (5.19) as RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_{\Gamma}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$ and the triplet $(F^{\mathrm{b}}_{\Gamma}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$ as its input data. Our specific choice of Γ will be $\Gamma = Y^{\mathrm{d}}_{\cdot}(\cdot)$ where $(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), M^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau))$ is a solution to the RBSDE$_{(\mathbb{G}^{\tau},\mathbb{P})}(F^{\mathrm{d}}D_{-}, S^{\mathrm{d}}(\tau)D, \xi^{\mathrm{d}}(\tau)D_T)$. In this case, we will show that a pasting of the resulting solutions of the two RBSDEs (5.17) and (5.19) yields a solution to the RBSDE(F, S, ξ), see Theorem 5.3.5.

To give a motivation for the choices of the alternative RBSDEs, we make use of the following proposition which gives the optional splitting formula of an $\mathcal{S}^p$-solution (Y, Z, U, M, K) to the RBSDE(F, S, ξ). We will denote by ζ the process defined for $t \in [0, T]$ by

$$\zeta \quad = \quad S_t 1_{\{t<T\}} + \xi 1_{\{t=T\}}. \tag{5.20}$$

Using the formulas (5.10) and (5.11) of ξ and S, we have $\zeta_t = \zeta^{\mathrm{b}} 1_{\{t<\tau\}} + \zeta^{\mathrm{d}}_t(\tau) 1_{\{t\geq\tau\}}$ where for $t \in [0, T]$

$$\zeta^{\mathrm{b}}_t = S^{\mathrm{b}}_t 1_{\{t<T\}} + \xi^{\mathrm{b}} 1_{\{t=T\}} \text{ and } \zeta^{\mathrm{d}}_t = S^{\mathrm{d}}_t 1_{\{t<T\}} + \xi^{\mathrm{d}} 1_{\{t=T\}}. \tag{5.21}$$

Proposition 5.3.4. *Let $p > 1$ and suppose that $\xi \in L^p(\mathcal{G}_T, \mathbb{P})$, $S \in \mathcal{S}^p_T(\mathbb{G}, \mathbb{P})$. Let (Y, Z, U, M, K) be an $\mathcal{S}^p$-solution to (5.13). Then there exists a quadruplet (Y^b, Z^b, M^b, K^b) of $\mathcal{O}(\mathbb{F})$-measurable processes and a quadruplet (Y^d, Z^d, M^d, K^d) of $\mathcal{O}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable processes such that for every $t \in [0, T]$*

$$\begin{cases} Y_t = Y^b_t 1_{\{t<\tau\}} + Y^d_t(\tau) 1_{\{t\geq\tau\}}, \\ Z_t = Z^b_t 1_{\{t\leq\tau\}} + Z^d_t(\tau) 1_{\{t>\tau\}}, \\ M_t = M^b_t 1_{\{t<\tau\}} + M^d_t(\tau) 1_{\{t\geq\tau\}}, \\ K_t = K^b_t 1_{\{t\leq\tau\}} + K^d_t(\tau) 1_{\{t>\tau\}}. \end{cases} \tag{5.22}$$

Moreover, $K^b_\tau 1_{\{\tau\leq T\}} = K^d_\tau(\tau) 1_{\{\tau\leq T\}}$ and $M^b_\tau 1_{\{\tau\leq T\}} = M^d_\tau(\tau) 1_{\{\tau\leq T\}}$.
Assume that $Y^d_{\cdot}(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable, then for Lebesgue-almost all $t \in [0, T]$

$$U_t = (Y^d_t(t) - Y^b_{t-}) 1_{\{t\geq\tau\}}. \tag{5.23}$$

Suppose additionally that $\int_0^T |F(s, Y_{s-}, Z_s, U_s)| ds \in L^p(\mathcal{G}_T, \mathbb{P})$. Then with ζ given by (5.20), we have for $t \in [0, T]$ and on $\{t \geq \tau\}$, we have

$$Y^d_t(\tau) \quad = \quad \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}(\mathbb{G}^\tau)} \mathbb{E}\left[\int_t^\sigma F^d(s, Y^d_{s-}(\tau), Z^d_s(\tau)) D_{s-} ds + \zeta^d_\sigma(\tau) D_\sigma \middle| \mathcal{G}^\tau_t\right]. \tag{5.24}$$

Moreover, with $\widehat{\mathbb{Q}}$ defined by (5.7) we have for $t \in [0, T]$

$$Y^b_t = \operatorname*{ess\,sup}_{\sigma^b\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[\int_t^{\sigma^b} \left[F^b(s, Y^b_{s-}, Z^b_s, Y^d_s(s) - Y^b_{s-}) + (Y^d_s(s) - Y^b_{s-})\lambda_s\right] ds + \zeta^b_{\sigma^b} \middle| \mathcal{F}_t\right]. \tag{5.25}$$

Proof. The existence of the quadruplets $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, M^{\mathrm{b}}, K^{\mathrm{b}})$ and $(Y^{\mathrm{d}}, Z^{\mathrm{d}}, M^{\mathrm{d}}, K^{\mathrm{d}})$ satisfying (5.22) is justified by Proposition 5.2.2. Since K is predictable and τ is totally inaccessible, K is continuous at τ. By definition of an $\mathcal{S}^p$-solution, $[M, N^{\mathbb{G}}] = 0$ which implies that

$$\left[M, N^{\mathbb{G}}\right]_T \quad = \quad \sum_{0<s\leq T} \Delta M_s \Delta N^{\mathbb{G}}_s = \Delta M_\tau 1_{\{\tau\leq T\}} = 0.$$

The equalities $K^{\mathrm{b}}_\tau 1_{\{\tau\leq T\}} = K^{\mathrm{d}}_\tau(\tau) 1_{\{\tau\leq T\}}$ and $M^{\mathrm{b}}_\tau 1_{\{\tau\leq T\}} = M^{\mathrm{d}}_\tau(\tau) 1_{\{\tau\leq T\}}$ are therefore a consequence of the splitting formulas of K and M given by (5.22).
Now assume that $Y^{\mathrm{d}}_{\cdot}(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable. Note that U is $\mathcal{P}(\mathbb{G})$-measurable. Thus there exists

a $\mathcal{P}(\mathbb{F})$-measurable process U^{b} and a $\mathcal{P}(\mathbb{F})\otimes\mathcal{B}(\mathbb{R}^+)$-measurable process U^{d} such that for every $t\in[0,T]$

$$U_t = U_t^{\mathrm{b}}1_{\{t\le\tau\}} + U_t^{\mathrm{d}}(\tau)1_{\{t>\tau\}} = U_t^{\mathrm{b}}(1-D_{t-}) + U_t^{\mathrm{d}}(\tau)D_{t-}.$$

It follows from decomposition (5.12) of F that $F(\cdot,Y_-,Z,U)$ and $F(\cdot,Y_-,Z,U^{\mathrm{b}}(1-D_-))$ are indistinguishable processes. As $N^{\mathbb{G}} = N^{\mathbb{G}}_{\cdot\wedge\tau}$ the processes $\int_0^\cdot U dN^{\mathbb{G}}$ and $\int_0^\cdot U^{\mathrm{b}}(1-D_-)dN^{\mathbb{G}}$ are indistinguishable. One can therefore assume w.l.o.g. that $U^{\mathrm{d}}=0$. Since $B^{\mathbb{G}},K$ and M are continuous at τ, we infer from the optional splitting formula of Y and its dynamical description that

$$\Delta Y_\tau 1_{\{\tau\le T\}} = (Y^{\mathrm{d}}_\tau(\tau) - Y^{\mathrm{b}}_{\tau-})1_{\{\tau\le T\}} = U^{\mathrm{b}}_\tau 1_{\{\tau\le T\}}.$$

Since $Y^{\mathrm{d}}_\cdot(\cdot), Y^{\mathrm{b}}_-$ and U^{b} are $\mathcal{O}(\mathbb{F})$-measurable, the above equality and Lemma 4.2.16 entail that for Lebesgue-almost all $t\in[0,T]$

$$U^{\mathrm{b}}_t \;=\; (Y^{\mathrm{d}}_t(t) - Y^{\mathrm{b}}_{t-})\ \mathbb{P}\text{-a.s.}.$$

Thus (5.23) holds.

We now suppose that $\mathbb{E}\left[\left(\int_0^T |F(s,Y_{s-},Z_s,U_s)|ds\right)^p\right] < +\infty$. Then Proposition 5.2.14 implies that

$$Y_\delta = \operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{\delta,T}(\mathbb{G})} \mathbb{E}\left[\int_\delta^\sigma F(s,Y_{s-},Z_s,U_s)ds + S_\sigma 1_{\{\sigma<T\}} + \xi 1_{\{\sigma=T\}}\middle|\mathcal{G}_\delta\right],\ \delta\in\mathcal{T}_T(\mathbb{G}). \tag{5.26}$$

Let $X = \zeta + \int_0^\cdot F(s,Y_{s-},Z_s,U_s)ds$. Due to our hypotheses on ξ and S, we have $X\in\mathcal{S}^p_T(\mathbb{G},\mathbb{P})$. Note that (5.26) entails that $V = Y + \int_0^\cdot F(s,Y_{s-},Z_s,U_s)ds$ is the $(\mathbb{G},\mathbb{P})$-Snell envelope of X. Using the formulas (5.21), (5.23) and the decomposition (5.12) of F, we have for $t\in[0,T]$

$$\begin{aligned}\int_0^t F(s,Y_{s-},Z_s,U_s)ds = &\left(\int_0^\tau F^{\mathrm{b}}(s,Y^{\mathrm{b}}_{s-},Z^{\mathrm{b}}_s,Y^{\mathrm{d}}_s(s)-Y^{\mathrm{b}}_{s-})ds + \int_0^t F^{\mathrm{d}}(s,Y^{\mathrm{d}}_{s-}(\tau),Z^{\mathrm{d}}_s(\tau))D_{s-}ds\right)1_{\{t\ge\tau\}}\\ &+\left(\int_0^t F^{\mathrm{b}}(s,Y^{\mathrm{b}}_{s-},Z^{\mathrm{b}}_s,Y^{\mathrm{d}}_s(s)-Y^{\mathrm{b}}_{s-})ds\right)1_{\{t<\tau\}}.\end{aligned}$$

Hence $X_t = X^{\mathrm{b}}_t 1_{\{t<\tau\}} + X^{\mathrm{d}}_t(\tau)1_{\{t\ge\tau\}}$ and $V_t = V^{\mathrm{b}}_t 1_{\{t<\tau\}} + V^{\mathrm{d}}_t(\tau)1_{\{t\ge\tau\}}$, where for $t\in[0,T]$

$$\begin{aligned}X^{\mathrm{b}}_t &= \zeta^{\mathrm{b}}_t + \int_0^t F^{\mathrm{b}}(s,Y^{\mathrm{b}}_{s-},Z^{\mathrm{b}}_s,Y^{\mathrm{d}}_s(s)-Y^{\mathrm{b}}_{s-})ds,\\ X^{\mathrm{d}}_t(\tau) &= \zeta^{\mathrm{d}}_t(\tau) + \int_0^\tau F^{\mathrm{b}}(s,Y^{\mathrm{b}}_{s-},Z^{\mathrm{b}}_s,Y^{\mathrm{d}}_s(s)-Y^{\mathrm{b}}_{s-})ds + \int_0^t F^{\mathrm{d}}(s,Y^{\mathrm{d}}_{s-}(\tau),Z^{\mathrm{d}}_s(\tau))D_{s-}ds,\\ V^{\mathrm{b}}_t &= Y^{\mathrm{b}}_t + \int_0^t F^{\mathrm{b}}(s,Y^{\mathrm{b}}_{s-},Z^{\mathrm{b}}_s,Y^{\mathrm{d}}_s(s)-Y^{\mathrm{b}}_{s-})ds,\\ V^{\mathrm{d}}_t(\tau) &= Y^{\mathrm{d}}_t(\tau) + \int_0^\tau F^{\mathrm{b}}(s,Y^{\mathrm{b}}_{s-},Z^{\mathrm{b}}_s,Y^{\mathrm{d}}_s(s)-Y^{\mathrm{b}}_{s-})ds + \int_0^t F^{\mathrm{d}}(s,Y^{\mathrm{d}}_{s-}(\tau),Z^{\mathrm{d}}_s(\tau))D_{s-}ds.\end{aligned}$$

As V is the Snell envelope of X, we deduce from Theorems 4.3.7 and 4.3.11 that for $t\in[0,T]$

$$\begin{aligned}V^{\mathrm{d}}_t(\tau)1_{\{t\ge\tau\}} &= \left(\operatorname*{ess\,sup}_{\sigma\in\mathcal{T}_{t,T}(\mathbb{G}^\tau)} \mathbb{E}\left[X^{\mathrm{d}}_\sigma(\tau)D_\sigma|\mathcal{G}^\tau_t\right]\right)1_{\{t\ge\tau\}},\\ V^{\mathrm{b}}_t &= \operatorname*{ess\,sup}_{\sigma^{\mathrm{b}}\in\mathcal{T}_{t,T}(\mathbb{F})} \mathbb{E}^{\widehat{\mathbb{Q}}}\left[X^{\mathrm{b}}_{\sigma^{\mathrm{b}}} + \int_t^{\sigma^{\mathrm{b}}}(V^{\mathrm{d}}_s(s)-V^{\mathrm{b}}_{s-})\lambda_s ds\middle|\mathcal{F}_t\right].\end{aligned}$$

The equality (5.24) results from the representations of $V^{\mathrm{d}}(\tau)$ and $X^{\mathrm{d}}(\tau)$. Regarding (5.25), it follows from the representations of $V^{\mathrm{b}}, X^{\mathrm{b}}$ and the equality $V^{\mathrm{d}}_t(t) - V^{\mathrm{b}}_t = Y^{\mathrm{d}}_t(t) - Y^{\mathrm{b}}_t$, $t\in[0,T]$. □

Proposition 5.3.4 gives an optional splitting formula of a solution (Y,Z,U,M,K) to the RBSDE(F,S,ξ) as well as the Snell envelope representation of the components appearing in the splitting formula of Y. By the classical relations between Snell's envelope and solutions to RBSDEs, (5.24) and (5.25) suggest a two steps algorithm to construct (Y,Z,U,M,K). Due to the dependence of Y^{b} on $Y_\cdot^{\mathrm{d}}(\cdot)$, to obtain (Y,Z,U,M,K), in a first step one looks for $\Big(Y^{\mathrm{d}}(\tau),Z^{\mathrm{d}}(\tau),M^{\mathrm{d}}(\tau),K^{\mathrm{d}}(\tau)\Big)$ and then in a second step for $\Big(Y^{\mathrm{b}},Z^{\mathrm{b}},M^{\mathrm{b}},K^{\mathrm{b}}\Big)$. Now (5.24) suggests that a solution to the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^{\mathrm{d}}D_-,S^{\mathrm{d}}(\tau)D,\xi^{\mathrm{d}}(\tau)D_T)$ is a good candidate for $\Big(Y^{\mathrm{d}}(\tau),Z^{\mathrm{d}}(\tau),M^{\mathrm{d}}(\tau),K^{\mathrm{d}}(\tau)\Big)$. Given the trace process $\Big\{Y_s^{\mathrm{d}}(s),s\in[0,T]\Big\}$, let $F^{\mathrm{b}}_{Y_\cdot^{\mathrm{d}}(\cdot)}$ be defined by

$$F^{\mathrm{b}}_{Y_\cdot^{\mathrm{d}}(\cdot)}(s,\cdot,y,z)=F^{\mathrm{b}}(s,\cdot,y,z,Y_s^{\mathrm{d}}(s)-y)+\lambda_s(Y_s^{\mathrm{d}}(s)-y),(s,y,z)\in[0,T]\times\mathbb{R}\times\mathbb{R}^n. \quad (5.27)$$

The representation (5.25) of Y^{b} suggests that one can look for $(Y^{\mathrm{b}},Z^{\mathrm{b}},M^{\mathrm{b}},K^{\mathrm{b}})$ as a solution to the RBSDE with data $(F^{\mathrm{b}}_{Y_\cdot^{\mathrm{d}}(\cdot)},S^{\mathrm{b}},\xi^{\mathrm{b}})$ in the filtration $\mathbb{F}$ and w.r.t. the measure $\widehat{\mathbb{Q}}$.

We are now ready to state the main result of this chapter which shows that the algorithm just described does lead to a solution to the RBSDE(F,S,ξ) provided the alternative RBSDEs admit a solution.

Theorem 5.3.5. *Let $p\geq 1$. Suppose that the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}$ $(F^dD_-,S^d(\tau)D,\xi^d(\tau)D)$ admits an $\mathcal{S}^p$-solution $(Y^d(\tau),Z^d(\tau),M^d(\tau),K^d(\tau))$ and the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}$ $(F^b_{Y_\cdot^d(\cdot)},S^b,\xi^b)$ admits an $\mathcal{S}^p$-solution (Y^b,Z^b,M^b,K^b). Assume moreover, that $Y_\cdot^d(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable. We consider (Y,Z,U,M,K) defined for $t\in[0,T]$ by*

$$\left\{\begin{array}{lcl} Y_t &=& Y_t^b 1_{\{t<\tau\}}+Y_t^d(\tau)1_{\{t\geq\tau\}},\\ Z_t &=& Z_t^b 1_{\{t\leq\tau\}}+Z_t^d(\tau)1_{\{t>\tau\}},\\ U_t &=& (Y_t^d(t)-Y_{t-}^b)1_{\{t\leq\tau\}},\\ M_t &=& M_{t\wedge\tau}^b+(M_t^d(\tau)-M_\tau^d(\tau))1_{\{t\geq\tau\}},\\ K_t &=& K_{t\wedge\tau}^b+(K_t^d(\tau)-K_\tau^d(\tau))1_{\{t>\tau\}}. \end{array}\right. \quad (5.28)$$

The quintuplet (Y,Z,U,M,K) is an $\mathcal{S}^p$-solution to the RBSDE(F,S,ξ).

The proof of the theorem relies on two lemmas on stochastic integration. In the sequel, for $\mathbb{H}\in\{\mathbb{F},\mathbb{G},\mathbb{G}^\tau\}$, X a real valued $\mathbb{F}$-semimartingale and $H\in L_{\mathbb{H}}(X)$, we use the notation $H\cdot_{\mathbb{H}}X$ for the stochastic integral of an $\mathbb{H}$-predictable process H w.r.t. X to emphasize the dependence on the filtration $\mathbb{H}$. We begin with the following stability result for stochastic integration w.r.t. different filtrations.

Lemma 5.3.6. *Let $X=(X_t)_{t\in[0,T]}$ be a continuous $\mathbb{F}$-semimartingale and $H\in L_{\mathbb{F}}(X)$. Then $H\in L_{\mathbb{G}}(X)\cap L_{\mathbb{G}^\tau}(X)$. Moreover, $H\cdot_{\mathbb{F}}X, H\cdot_{\mathbb{G}}X$ and $H\cdot_{\mathbb{G}^\tau}X$ are indistinguishable.*

Proof. As X is a continuous semimartingale, it admits the decomposition $X=M^X+A^X$ where M^X is a continuous $(\mathbb{F},\mathbb{P})$-local martingale and A^X an $\mathbb{F}$-predictable process of finite variation. Since $H\in L_{\mathbb{F}}(X)$ and X is continuous, $H\cdot_{\mathbb{F}}X$ is an $(\mathbb{F},\mathbb{P})$-special semimartingale. Therefore Theorem 2 in [CMS80] implies that $H\in L_{\mathbb{F}}(M^X)\cap L_{\mathbb{F}}(A^X)$. Moreover, $H\cdot_{\mathbb{F}}A^X$ exists as a Stieltjes integral and

$$H\cdot_{\mathbb{F}}X \quad = \quad H\cdot_{\mathbb{F}}M^X+H\cdot_{\mathbb{F}}A^X.$$

Since $H\cdot_{\mathbb{F}}A^X$ exists as a Stieltjes integral, $H\in L_{\mathbb{G}}(A^X)$. To show that $H\in L_{\mathbb{G}}(X)$, it remains to show that $H\in L_{\mathbb{G}}(M^X)$. Due to the density hypothesis, and the continuity of M^X, the processes M^X and $H\cdot_{\mathbb{F}}M^X$ are special semimartingales w.r.t to $\mathbb{G}$. Let $M^X=M^{\mathbb{G}}+A^{\mathbb{G}}$ be the canonical decomposition of M^X in $\mathbb{G}$. As $H\in L_{\mathbb{F}}(M^X)$ and M^X is a continuous $(\mathbb{F},\mathbb{P})$-local martingale, $(\int_0^t H_s^2d[M^X,M^X]_s)_{t\in[0,T]}$ is locally integrable. The process $H\cdot_{\mathbb{F}}M^X$ being a $\mathbb{G}$-semimartingale, we infer from [Pro04, Theorem 5, VI] that $\int_0^{\cdot} HdA^{\mathbb{G}}$ exists as a path-by-path Lebesgue Stieltjes integral and thus $H \in L_{\mathbb{G}}(A^{\mathbb{G}})$. Clearly[3] $H \in L_{\mathbb{G}}(M^{\mathbb{G}})$ and therefore $H \in L_{\mathbb{G}}(M^{\mathbb{G}}) \cap L_{\mathbb{G}}(A^{\mathbb{G}}) = L_{\mathbb{G}}(M^X)$. We conclude that $H \in L_{\mathbb{G}}(X) \cap L_{\mathbb{F}}(X)$. Since $\mathbb{F} \subseteq \mathbb{G}$, Theorem 7 in [Jac80] or [HWY92, Theorem 12.37] implies that the stochastic integral processes $H \cdot_{\mathbb{F}} X$ and $H \cdot_{\mathbb{G}} X$ are indistinguishable. We use similar arguments to show that $H \in L_{\mathbb{G}^\tau}(X)$ and that the processes $H \cdot_{\mathbb{F}} X$ and $H \cdot_{\mathbb{G}^\tau} X$ are indistinguishable. □

Remark 5.3.7. *Lemma 5.3.6 remains true if H is locally bounded and X is càdlàg as shown in[Pro04, Theorem 33, IV] for arbitrary filtrations $\mathbb{F}, \mathbb{G}$ satisfying $\mathbb{F} \subseteq \mathbb{G}$.*

The following lemma gives a useful decomposition of the Brownian motion $B^{\mathbb{G}}$ and a formula for the computation of stochastic integrals w.r.t. $B^{\mathbb{G}}$. We recall from (5.9) and Remark 5.3.1 that

$$B_t^{\mathbb{G}} \;=\; B_t - \int_0^{t\wedge\tau} \frac{d\langle B, L^{\mathbb{F}}\rangle_s}{L^{\mathbb{F}}_{s-}} - \int_{t\wedge\tau}^t \frac{d\langle B, \alpha^{\mathrm{d}}(u)\rangle}{\alpha^{\mathrm{d}}_{s-}(u)}\bigg|_{u=\tau} \tag{5.29}$$

Lemma 5.3.8. *Let $B^{\mathbb{F}}$ and $B^{\mathbb{G}^\tau}$ be given by (5.15) and (5.16). For $i = 1, 2, \cdots n$, let $\overline{B}^{\mathbb{F},i} = (1 - D_-) \cdot_{\mathbb{G}} B^{\mathbb{F},i}$ and $\overline{B}^{\mathbb{G}^\tau,i} = D_- \cdot_{\mathbb{G}^\tau} B^{\mathbb{G}^\tau,i}$ be the processes defined for $t \in [0,T]$ by*

$$\overline{B}_t^{\mathbb{F},i} \;=\; B_t^{\mathbb{F},i} 1_{\{t<\tau\}} + B_\tau^{\mathbb{F},i} 1_{\{t\geq\tau\}} = B^{\mathbb{F},i}_{t\wedge\tau} = \int_0^t (1 - D_{s-}) dB_s^{\mathbb{F},i}, \tag{5.30}$$

$$\overline{B}_t^{\mathbb{G}^\tau,i} \;=\; (B_t^{\mathbb{G}^\tau,i} - B_\tau^{\mathbb{G}^\tau,i}) 1_{\{t\geq\tau\}} = \int_0^t D_{s-} dB_s^{\mathbb{G}^\tau,i}. \tag{5.31}$$

Let $i \in \{1, 2, \cdots, n\}$. The processes $\overline{B}^{\mathbb{F},i}$ and $\overline{B}^{\mathbb{G}^\tau,i}$ are continuous $(\mathbb{G}, \mathbb{P})$-local martingales. Moreover,

$$B^{\mathbb{G},i} = \overline{B}^{\mathbb{F},i} + \overline{B}^{\mathbb{G}^\tau,i}. \tag{5.32}$$

Let $Z^{b,i} \in L_{\mathbb{F}}(B^{\mathbb{F},i})$. Let $Z^{d,i}$ be a $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}^+)$-measurable process satisfying $Z^{d,i}(\tau) \in L_{\mathbb{G}^\tau}(B^{\mathbb{G}^\tau,i})$. The following hold:

a) $Z^{b,i}(1 - D_-) \in L_{\mathbb{G}}(B^{\mathbb{F},i}) \cap L_{\mathbb{G}}(\overline{B}^{\mathbb{F},i}) \cap L_{\mathbb{G}}(B^{\mathbb{G},i})$ and

$$\int_0^{\cdot} Z_s^{b,i}(1 - D_{s-}) dB_s^{\mathbb{F},i} \;=\; \int_0^{\cdot} Z_s^{b,i}(1 - D_{s-}) d\overline{B}_s^{\mathbb{F},i} = \int_0^{\cdot} Z_s^b (1 - D_{s-}) dB^{\mathbb{G},i}. \tag{5.33}$$

b) $Z^{d,i}(\tau) D_- \in L_{\mathbb{G}^\tau}(B^{\mathbb{G}^\tau,i}) \cap L_{\mathbb{G}}(\overline{B}^{\mathbb{G}^\tau,i}) \cap L_{\mathbb{G}}(B^{\mathbb{G},i})$. Moreover,

$$\int_0^{\cdot} Z_s^{d,i}(\tau) D_{s-} dB_s^{\mathbb{G}^\tau,i}(\tau) \;=\; \int_0^{\cdot} Z_s^{d,i}(\tau) D_{s-} d\overline{B}_s^{\mathbb{G}^\tau,i} = \int_0^{\cdot} Z_s^{d,i}(\tau) D_{s-} dB_s^{\mathbb{G},i}. \tag{5.34}$$

c) Let $Z^i = Z^{b,i}(1 - D_-) + Z^{d,i}(\tau) D_-$. Then $Z^i \in L_{\mathbb{G}}(B^{\mathbb{G},i})$ and

$$\int_0^{\cdot} Z_s^i dB_s^{\mathbb{G},i} \;=\; \int_0^{\cdot} Z_s^{b,i}(1 - D_{s-}) dB_s^{\mathbb{F},i} + \int_0^{\cdot} Z_s^{d,i}(\tau) D_{s-} dB_s^{\mathbb{G}^\tau,i}. \tag{5.35}$$

Proof. Let $i \in \{1, 2, \cdots, n\}$. Let $t \in [0, T]$. Using (5.29) and properties of integration, we have

$$\begin{aligned}
B_t^{\mathbb{G},i} \;&=\; B_t^i - \int_0^{t\wedge\tau} \frac{d\langle B^i, L^{\mathbb{F}}\rangle_s}{L^{\mathbb{F}}_{s-}} - \int_0^t \frac{d\langle B^i, \alpha^{\mathrm{d}}(u)\rangle_s}{\alpha^{\mathrm{d}}_{s-}(u)}\bigg|_{u=\tau} 1_{\{t\geq\tau\}} + \int_0^\tau \frac{d\langle B^i, \alpha^{\mathrm{d}}(u)\rangle_s}{\alpha^{\mathrm{d}}_{s-}(u)}\bigg|_{u=\tau} 1_{\{t\geq\tau\}} \\
&=\; \left(B_t^i - \int_0^t \frac{d\langle B^i, L^{\mathbb{F}}\rangle_s}{L^{\mathbb{F}}_{s-}}\right) 1_{\{t<\tau\}} + \left(B_t^i - \int_0^t \frac{d\langle B^i, \alpha^{\mathrm{d}}(u)\rangle_s}{\alpha^{\mathrm{d}}_{s-}(u)}\bigg|_{u=\tau}\right) 1_{\{t\geq\tau\}} \\
&\quad + \left(B_\tau^i - \int_0^\tau \frac{d\langle B^i, L^{\mathbb{F}}\rangle_s}{L^{\mathbb{F}}_{s-}}\right) 1_{\{t\geq\tau\}} + \left(-B_\tau^i + \int_0^\tau \frac{d\langle B^i, \alpha^{\mathrm{d}}(u)\rangle_s}{\alpha^{\mathrm{d}}_{s-}(u)}\bigg|_{u=\tau}\right) 1_{\{t\geq\tau\}} \\
&=\; B_t^{\mathbb{F},i} 1_{\{t<\tau\}} + B_\tau^{\mathbb{F},i} 1_{\{t\geq\tau\}} + (B_t^{\mathbb{G}^\tau,i} - B_\tau^{\mathbb{G}^\tau,i}) 1_{\{t\geq\tau\}} = \overline{B}_t^{\mathbb{F},i} + \overline{B}_t^{\mathbb{G}^\tau,i}.
\end{aligned}$$

[3] This is due to the fact that $(\int_0^t H_s^2 d[M^{\mathbb{G}}, M^{\mathbb{G}}]_s)_{t\in[0,T]} = (\int_0^t H_s^2 d[M^X, M^X]_s)_{t\in[0,T]}$ is locally integrable.

The continuity of $\overline{B}^{\mathbb{F},i}$ and $\overline{B}^{\mathbb{G}^\tau,i}$ follows from the continuity of $B^{\mathbb{F},i}$ and $B^{\mathbb{G}^\tau,i}$. By Remark 5.3.1, $B^{\mathbb{F}}L^{\mathbb{F}}$ is an $(\mathbb{F},\mathbb{P})$-local martingale. We infer from [EKJJ10, Proposition 5.1] that the stopped process $\overline{B}^{\mathbb{F},i}$ is a $(\mathbb{G},\mathbb{P})$-local martingale. As $B^{\mathbb{G}^\tau,i}$ is a $(\mathbb{G}^\tau,\mathbb{P})$-local martingale, $\overline{B}^{\mathbb{G}^\tau,i} = D_- \cdot_{\mathbb{G}^\tau} B^{\mathbb{G}^\tau,i}$ is a $(\mathbb{G}^\tau,\mathbb{P})$-local martingale.

a) By Lemma 5.3.6, $Z^{b,i} \in L_{\mathbb{G}}(B^{\mathbb{F},i})$. Note that $1-D_-$ is predictable, bounded and $(1-D_-)^2 = 1-D_-$. Hence $(1-D_-)^2 \in L_{\mathbb{G}}(Z^{b,i} \cdot_{\mathbb{G}} B^{\mathbb{F},i})$. Consequently, $Z^{b,i}(1-D_-)^2 \in L_{\mathbb{G}}(B^{\mathbb{F},i})$ and the associativity property of the stochastic integral (see [Pro04, Theorem 21, IV]) entails that $Z^{b,i}(1-D_-) \in L_{\mathbb{G}}(B^{\mathbb{F},i}) \cap L_{\mathbb{G}}((1-D_-)\cdot_{\mathbb{G}} B^{\mathbb{F},i}) = L_{\mathbb{G}}(B^{\mathbb{F},i}) \cap L_{\mathbb{G}}(\overline{B}^{\mathbb{F},i})$. Moreover,

$$\int_0^\cdot Z_s^{b,i}(1-D_{s-})dB_s^{\mathbb{F},i} \;=\; \int_0^\cdot Z_s^{b,i}(1-D_{s-})^2 dB_s^{\mathbb{F},i} = \int_0^\cdot Z_s^{b,i}(1-D_{s-})d\overline{B}_s^{\mathbb{F},i}. \tag{5.36}$$

To show that $Z^{b,i}(1-D_-) \in L_{\mathbb{G}}(B^{\mathbb{G},i})$, we show that $Z^{b,i}(1-D_-) \in L_{\mathbb{G}}(\overline{B}^{\mathbb{G}^\tau,i})$ since $B^{\mathbb{G},i} = \overline{B}^{\mathbb{F},i} + \overline{B}^{\mathbb{G}^\tau,i}$ and $Z^{b,i}(1-D_-) \in L_{\mathbb{G}}(\overline{B}^{\mathbb{F},i})$. To this end, note that $\overline{B}^{\mathbb{G}^\tau,i}$ is a $(\mathbb{G},\mathbb{P})$-local martingale with quadratic variation $[\overline{B}^{\mathbb{G}^\tau,i}] = \int_0^\cdot D_{s-}^2 d[B^{\mathbb{G}^\tau,i}]_s$. Using the equality $(1-D_-)D_- = 0$, we have

$$\int_0^\cdot |Z_s^{b,i}(1-D_{s-})|^2 d[\overline{B}^{\mathbb{G}^\tau,i}]_s = \int_0^\cdot |Z_s^{b,i}(1-D_{s-})|^2 D_{s-} d[B^{\mathbb{G}^\tau,i}]_s = 0.$$

We deduce that $Z^{b,i}(1-D_-) \in L_{\mathbb{G}}(\overline{B}^{\mathbb{G}^\tau,i})$. Furthermore, $\int_0^\cdot Z_s^{b,i}(1-D_{s-})d\overline{B}_s^{\mathbb{G}^\tau,i}$ is a continuous local martingale with quadratic variation zero and thus a null martingale. It follows from (5.36) and the definition of $B^{\mathbb{G},i}$ that

$$\int_0^\cdot Z_s^{b,i}(1-D_{s-})dB_s^{\mathbb{G},i} \;=\; \int_0^\cdot Z_s^{b,i}(1-D_{s-})d\overline{B}_s^{\mathbb{F},i}. \tag{5.37}$$

b) First note that $Z^{d,i}(\tau)D_-$ is $\mathcal{P}(\mathbb{G})$- and $\mathcal{P}(\mathbb{G}^\tau)$-measurable by Proposition 5.2.2. As $Z^{\mathrm{d},i}(\tau) \in L_{\mathbb{G}^\tau}(B^{\mathbb{G}^\tau,i}(\tau))$, D_- is bounded and $D_-^2 = D_-$, we have $Z^{\mathrm{d},i}(\tau)D_- = Z^{\mathrm{d},i}(\tau)D_-D_- \in L_{\mathbb{G}^\tau}(B^{\mathbb{G}^\tau,i})$. We infer from [Pro04, Theorem 21,IV] that $Z^{d,i}(\tau)D_- \in L_{\mathbb{G}^\tau}(D_- \cdot_{\mathbb{G}^\tau} B^{\mathbb{G}^\tau,i}) = L_{\mathbb{G}^\tau}(\overline{B}^{\mathbb{G}^\tau,i})$ and

$$\int_0^\cdot Z_s^{\mathrm{d},i}(\tau)D_- dB_s^{\mathbb{G}^\tau,i} \;=\; \int_0^\cdot Z_s^{\mathrm{d},i}(\tau)D_{s-}D_{s-}dB_s^{\mathbb{G}^\tau,i} = \int_0^\cdot Z_s^{\mathrm{d},i}(\tau)D_{s-}d\overline{B}_s^{\mathbb{G}^\tau,i}.$$

The process $\overline{B}^{\mathbb{G}^\tau,i}$ being a $\mathbb{G}$ and $\mathbb{G}^\tau$-semimartingale, and $Z^{d,i}(\tau)D_- \in L_{\mathbb{G}^\tau}(\overline{B}^{\mathbb{G}^\tau,i})$, Theorem 7 in [Jac80] implies that $Z^{d,i}(\tau)D_- \in L_{\mathbb{G}}(\overline{B}^{\mathbb{G}^\tau,i})$. In addition, the stochastic integral processes $Z^{d,i}(\tau)D_- \cdot_{\mathbb{G}^\tau} \overline{B}^{\mathbb{G}^\tau,i}$ and $Z^{d,i}(\tau)D_- \cdot_{\mathbb{G}} \overline{B}^{\mathbb{G}^\tau,i}$ are indistinguishable. One verifies that $[\overline{B}^{\mathbb{F},i}] = \int_0^\cdot (1-D_{s-})d[B^{\mathbb{F},i}]_s$. Using once more the equality $(1-D_-)D_- = 0$, we see that

$$\int_0^\cdot |Z_s^{d,i}(\tau)D_{s-}|^2 d[\overline{B}^{\mathbb{F},i}]_s = \int_0^\cdot |Z_s^{d,i}(\tau)D_{s-}|^2(1-D_{s-})d[B^{\mathbb{F},i}]_s = 0.$$

The last equality implies that $Z^{d,i}(\tau)D_- \in L_{\mathbb{G}}(\overline{B}^{\mathbb{G}^\tau,i})$ and $Z^{d,i}(\tau)D_- \cdot_{\mathbb{G}} \overline{B}^{\mathbb{G}^\tau,i}$ has quadratic variation 0. Due the continuity of $\overline{B}^{\mathbb{G}^\tau,i}$, we have $Z^{d,i}(\tau)D_- \cdot_{\mathbb{G}} \overline{B}^{\mathbb{G}^\tau,i} = 0$. Hence $Z^{d,i}(\tau)D_- \in L_{\mathbb{G}}(B^{\mathbb{G}})$ and (5.34) holds.

c)This follows from a), b) and the additivity property of the stochastic integral. □

Proof of Theorem 5.3.5. We will carry out the proof in three steps.
Step 1. We show that $(Y,Z,U,M,K) \in \mathcal{S}^p_{\mathbf{sol}}(\mathbb{G},\mathbb{P})$. Clearly K is increasing. Due to the measurability and integrability properties of $Y^{\mathrm{b}}, Y^{\mathrm{d}}(\tau), Z^{\mathrm{b}}, Z^{\mathrm{d}}(\tau), K^{\mathrm{b}}$ and $K^{\mathrm{d}}(\tau)$, Proposition 5.2.2 and Lemma 5.2.8 imply that $(Y,Z,K) \in \mathcal{S}^p_T(\mathbb{G},\mathbb{P}) \times \mathcal{H}^{n,p}_T(\mathbb{G},\mathbb{P}) \times \mathcal{IS}^p(\mathbb{G},\mathbb{P})$. Let us now show that

- $M \in \mathcal{M}^p_{\mathbb{G}}(B^{\mathbb{G}}, \mathbb{P}) \cap \mathcal{M}^p_{\mathbb{G}}(N^{\mathbb{G}}, \mathbb{P})$. First note that $M = \overline{M}^{\mathrm{b}} + \overline{M}^{\mathrm{d}}(\tau)$, where for $t \in [0,T]$ we have

$$
\begin{aligned}
\overline{M}^{\mathrm{b}}_t &= M^{\mathrm{b}}_t 1_{\{t<\tau\}} + M^{\mathrm{b}}_\tau 1_{\{t\geq\tau\}} = M^{\mathrm{b}}_{t\wedge\tau} = \left((1-D_-)\cdot_{\mathbb{G}} M^{\mathrm{b}}\right)_t = \int_0^t (1-D_{s-})dM^{\mathrm{b}}_s,\\
\overline{M}^{\mathrm{d}}_t(\tau) &= (M^{\mathrm{d}}_t(\tau) - M^{\mathrm{d}}_\tau(\tau))1_{\{t\geq\tau\}} = \left(D_- \cdot_{\mathbb{G}^\tau} M^{\mathrm{d}}(\tau)\right)_t = \int_0^t D_{s-}dM^{\mathrm{d}}_s(\tau).
\end{aligned}
$$

As $M^{\mathrm{b}} \in M^p_{\mathbb{F}}(\mathbb{F}, \mathbb{Q})$, $M^{\mathrm{b}} L^{\mathbb{F}}$ is a $\mathbb{F}$-local martingale. We infer from [EKJJ10, Proposition 5.1] that $\overline{M}^{\mathrm{b}}$ is a $(\mathbb{G}, \mathbb{P})$-local martingale. Since $M^{\mathrm{d}}(\tau) \in \mathcal{M}^p_{\mathbb{G}^\tau}(\mathbb{G}^\tau, \mathbb{P})$, we have $\overline{M}^{\mathrm{d}}(\tau) = D_- \cdot_{\mathbb{G}^\tau} M^{\mathrm{d}}(\tau)$ is a $(\mathbb{G}^\tau, \mathbb{P})$-local martingale. We deduce that M is a $(\mathbb{G}, \mathbb{P})$-local martingale as the sum of two $(\mathbb{G}, \mathbb{P})$-local martingales.
We now show that $\mathbb{E}\left[[M]^{\frac{p}{2}}_T\right] < +\infty$. Observe that for $t \in [0,T]$, we have $\left[\overline{M}\right]_t = [M^{\mathrm{b}}]_{t\wedge\tau}$ and $[\overline{M}^{\mathrm{d}}(\tau)]_t = \left([M^{\mathrm{d}}(\tau)]_t - [M^{\mathrm{d}}(\tau)]_\tau\right)1_{\{t\geq\tau\}}$. Using the inequality of Kunita-Watanabe and the binomial inequalities, we have

$$
\begin{aligned}
[M]_T &= [\overline{M}^{\mathrm{b}} + \overline{M}^{\mathrm{d}}(\tau)]_T \leq 2[\overline{M}^{\mathrm{b}}]_T + 2[\overline{M}^{\mathrm{d}}(\tau)]_T\\
&\leq 2[M^{\mathrm{b}}]_{T\wedge\tau} + 2\left([M^{\mathrm{d}}(\tau)]_T - [M^{\mathrm{d}}(\tau)]_\tau\right)1_{\{T\geq\tau\}} \leq 2[M^{\mathrm{b}}]_{T\wedge\tau} + 2[M^{\mathrm{d}}(\tau)]_T. \quad (5.38)
\end{aligned}
$$

By hypothesis, $\mathbb{E}^{\widehat{\mathbb{Q}}}\left[[M^{\mathrm{b}}]^{\frac{p}{2}}_T\right] + \mathbb{E}\left[[M^{\mathrm{d}}(\tau)]^{\frac{p}{2}}_T\right] < +\infty$. Applying Lemma 5.2.8, we see that

$$
\mathbb{E}\left[[M^{\mathrm{b}}]^{\frac{p}{2}}_{T\wedge\tau} + [M^{\mathrm{d}}(\tau)]^{\frac{p}{2}}_T\right] = \mathbb{E}\left[[M^{\mathrm{b}}]^{\frac{p}{2}}_T 1_{\{T<\tau\}} + [M^{\mathrm{b}}]^{\frac{p}{2}}_\tau 1_{\{\tau\leq T\}} + [M^{\mathrm{d}}(\tau)]^{\frac{p}{2}}_T\right] < +\infty. \quad (5.39)
$$

Using the binomial inequality $(a+b)^{\frac{p}{2}} \leq 2^{\frac{p}{2}}(a^{\frac{p}{2}} + b^{\frac{p}{2}})$ for $a, b \in \mathbb{R}^+$, we see that (5.38) and (5.39) lead to $\mathbb{E}\left[[M]^{\frac{p}{2}}_T\right] < +\infty$.
Note that M is continuous at τ. Since $N^{\mathbb{G}}$ is a quadratic pure jump martingale, we have

$$
[M, N^{\mathbb{G}}]_t = \sum_{0<s\leq t} \Delta M_s \Delta N^{\mathbb{G}}_s = \Delta M_\tau 1_{\{t\geq\tau\}} = 0, \ t \in [0,T].
$$

It remains to show that $[M, B^{\mathbb{G},i}] = 0$ for $i = 1, 2, \cdots, n$. Fix $i \in \{1, 2, \cdots, n\}$. Recalling that $M = \overline{M}^{\mathrm{b}} + \overline{M}^{\mathrm{d}}(\tau)$ and $B^{\mathbb{G},i} = \overline{B}^{\mathbb{F},i} + \overline{B}^{\mathbb{G}^\tau,i}$ by Lemma 5.3.8. Hence $[M, B^{\mathbb{G},i}]$ has the form

$$
[M, B^{\mathbb{G},i}] = [\overline{M}^{\mathrm{b}}, \overline{B}^{\mathbb{F},i}] + [\overline{M}^{\mathrm{b}}, \overline{B}^{\mathbb{G}^\tau,i}] + [\overline{M}^{\mathrm{d}}(\tau), \overline{B}^{\mathbb{F},i}] + [\overline{M}^{\mathrm{d}}(\tau), \overline{B}^{\mathbb{G}^\tau,i}]. \quad (5.40)
$$

Note that $(1-D_-)D_- = 0$. We infer from the definitions of $\overline{M}^{\mathrm{b}}, \overline{M}^{\mathrm{d}}(\tau)$, (5.30) and (5.31) that

$$
\begin{aligned}
[\overline{M}^{\mathrm{b}}, B^{\mathbb{G}^\tau,i}] &= \int_0^\cdot (1-D_{s-})D_{s-}d[M^{\mathrm{b}}, B^{\mathbb{G}^\tau,i}]_s = 0,\\
[\overline{M}^{\mathrm{d}}(\tau), \overline{B}^{\mathbb{F},i}] &= \int_0^\cdot (1-D_{s-})D_{s-}d[B^{\mathbb{G}^\tau,i}, B^{\mathbb{F},i}]_s = 0.
\end{aligned}
$$

By definition of solutions, $[M^{\mathrm{b}}, B^{\mathbb{F},i}] = 0$ and $[M^{\mathrm{d}}(\tau), B^{\mathbb{G}^\tau,i}] = 0$. As a result, we have

$$
[\overline{M}^{\mathrm{b}}, \overline{B}^{\mathbb{F},i}] = \int_0^\cdot (1-D_{s-})d[M^{\mathrm{b}}, B^{\mathbb{F},i}]_s = 0, \ [\overline{M}^{\mathrm{d}}(\tau), \overline{B}^{\mathbb{G}^\tau,i}] = \int_0^\cdot D_{s-}d[M^{\mathrm{d}}(\tau), B^{\mathbb{G}^\tau,i}]_s = 0.
$$

We deduce from (5.40) that $[M, B^{\mathbb{G},i}] = 0$. We therefore have $M \in \mathcal{M}^p_{\mathbb{G}}(B^{\mathbb{G}}, \mathbb{P}) \cap \mathcal{M}^p_{\mathbb{G}}(N^{\mathbb{G}}, \mathbb{P})$.

- We show that $U \in \mathcal{L}_T^p(\mathbb{G},\mathbb{P})$. Relying on the integrability properties of $Y^{\rm d}(\tau)$ and $Y^{\rm b}$, Lemma 5.2.8 implies that $\mathbb{E}\left[|Y_\tau^{\rm d}(\tau)-Y_{\tau-}^{\rm b}|^p 1_{\{\tau\le T\}}\right]<+\infty$. Consequently,

$$\mathbb{E}\left[\left(\int_0^T U_s^2 dD_s\right)^{\frac{p}{2}}\right]=\mathbb{E}\left[|U_\tau|^p D_T\right]=\mathbb{E}\left[|Y_\tau^{\rm d}(\tau)-Y_{\tau-}^{\rm b}|^p 1_{\{\tau\le T\}}\right]<+\infty.$$

Step 2. We derive an equation for the dynamics of Y and show that it satisfies (5.13). To this end, we will rely on Itô's formula and the equations describing the dynamics for $Y^{\rm b}$ and $Y^{\rm d}(\tau)$. First we derive an equation for $Y^{\rm b}(1-D)$. We recall from (5.19) that for $t\in[0,T]$

$$dY_t^{\rm b} = -F^{\rm b}(t,Y_{t-}^{\rm b},Z_t^{\rm b},Y_t^{\rm d}(t)-Y_{t-}^{\rm b})dt-(Y_t^{\rm d}(t)-Y_{t-}^{\rm b})\lambda_t dt-dK_t^{\rm b}+Z_t^{\rm b}dB_t^{\mathbb{F}}+dM_t^{\rm b} \tag{5.41}$$

Note that $Y^{\rm b}$ is $\mathbb{F}$-adapted and therefore continuous at τ. Since D is a quadratic pure jump semimartingale[4], we have $[Y^{\rm b},1-D]=0$. We deduce from Itô's formula that for $t\in[0,T]$

$$\begin{aligned}
d(Y_t^{\rm b}1_{t<\tau}) &= d(Y_t^{\rm b}(1-D_t))=-Y_{t-}^{\rm b}dD_t+(1-D_{t-})dY_t^{\rm b}\\
&= -Y_{t-}^{\rm b}dD_t+(1-D_{t-})\big(-F^{\rm b}(t,Y_{t-}^{\rm b},Z_t^{\rm b},Y_t^{\rm d}(t)-Y_{t-}^{\rm b})dt-(Y_t^{\rm d}(t)-Y_{t-}^{\rm b})\lambda_t dt\big)\\
&\quad +(1-D_{t-})\big(-dK_t^{\rm b}+Z^{\rm b}dB_t^{\mathbb{F}}+dM_t^{\rm b}\big).
\end{aligned}$$

Next we look at the equation satisfied by $Y^{\rm d}(\tau)D$. Recall that for $t\in[0,T]$

$$\begin{aligned}
dY_t^{\rm d}(\tau) &= -F^{\rm d}(t,Y_{t-}^{\rm d}(\tau),Z_t^{\rm d}(\tau))D_{t-}dt-dK_t^{\rm d}(\tau)+Z_t^{\rm d}(\tau)dB_t^{\mathbb{G}^\tau}+dM_t^{\rm d}(\tau),\\
d(Y_t^{\rm d}(\tau)D_t) &= D_{t-}dY_t^{\rm d}(\tau)+Y_{t-}^{\rm d}(\tau)dD_t+d\big[Y^{\rm d}(\tau),D\big]_t.
\end{aligned} \tag{5.42}$$

As D is a quadratic pure jump semimartingale, we have

$$[Y^{\rm d}(\tau),D]_t = \sum_{0<s\le t}\Delta Y_s^{\rm d}(\tau)\Delta D_s=(Y_\tau^{\rm d}(\tau)-Y_{\tau-}^{\rm d}(\tau))D_t=\int_0^t(Y_s^{\rm d}(s)-Y_{s-}^{\rm d}(s))dD_s.$$

The above equality and the identity $\int_0^t Y_{s-}^{\rm d}(\tau)dD_s=Y_{\tau-}^{\rm d}(\tau)D_t=\int_0^t Y_{s-}^{\rm d}(s)dD_s$ lead to

$$Y_{t-}^{\rm d}(\tau)dD_t+d\big[Y^{\rm d}(\tau),D\big]_t=Y_t^{\rm d}(t)dD_t.$$

Inserting the latter equality into (5.42), we obtain that for $t\in[0,T]$

$$\begin{aligned}
d(Y_t^{\rm d}(\tau)D_t) &= D_{t-}\big(-F^{\rm d}(t,Y_{t-}^{\rm d}(\tau),Z_t^{\rm d}(\tau))1_{\{t>\tau\}}dt-dK_t^{\rm d}(\tau)+Z_t^{\rm d}(\tau)dB_t^{\mathbb{G}^\tau}+dM_t^{\rm d}(\tau)\big)\\
&\quad +Y_t^{\rm d}(t)dD_t.
\end{aligned}$$

Combining the equations for $Y^{\rm b}(1-D)$ and $Y^{\rm d}(\tau)D$ yields for $t\in[0,T]$

$$\begin{aligned}
dY_t &= d(Y_t^{\rm b}(1-D_t))+d(Y_t^{\rm d}(\tau)D_t)\\
&= -(1-D_{t-})F^{\rm b}(t,Y_{t-}^{\rm b},Z_t^{\rm b},Y_t^{\rm d}(t)-Y_{t-}^{\rm b})dt-D_{t-}F^{\rm d}(t,Y_{t-}^{\rm d}(\tau),Z_t^{\rm d}(\tau))dt\\
&\quad -(1-D_{t-})dK_t^{\rm b}-D_{t-}dK_t^{\rm d}(\tau)\\
&\quad +(Y_t^{\rm d}(t)-Y_{t-}^{\rm b})dD_t-(1-D_{t-})(Y_t^{\rm d}(t)-Y_{t-}^{\rm b})\lambda_t dt\\
&\quad (1-D_{t-})Z_t^{\rm b}dB_t^{\mathbb{F}}+D_{t-}Z_t^{\rm d}(\tau)dB_t^{\mathbb{G}^\tau}+(1-D_{t-})dM_t^{\rm b}+D_{t-}dM_{t-}^{\rm d}(\tau).
\end{aligned}$$

We recall that $U=(Y_\cdot^{\rm d}(\cdot)-Y_-^{\rm b})(1-D_-)$ and $D_t=1_{\{t\ge\tau\}},\ t\in[0,T]$. Since $Y^{\rm b}\in\mathcal{S}_T^p(\mathbb{F},\mathbb{Q})$ and $Y^{\rm d}(\tau)\in\mathcal{S}_T^p(\mathbb{G}^\tau,\mathbb{P})$, we have $\int_0^\cdot(Y_\cdot^{\rm d}(\cdot)-Y_{\cdot-}^{\rm b})dD=\int_0^\cdot U_s dD_s$ is a $\mathbb{G}$-special semimartingale [5]. As

[4] A semimartingale X is said to be of *quadratic pure jump* if its quadratic variation process is purely discontinuous, i.e., $[X,X]^c=0$.

[5] For $t\in[0,T]$, set $\Pi_t=\int_0^t U_s dD_s=U_\tau D_t=(Y_\tau^{\rm d}(\tau)-Y_\tau^{\rm b})1_{\{t\ge\tau\}}, t\in[0,T]$. We have $\mathbb{E}\left[\sup_{t\in[0,T]}|\Pi_t|\right]=\mathbb{E}\left[|Y_\tau^{\rm d}(\tau)-Y_\tau^{\rm b}|1_{\{\tau\le T\}}\right]<+\infty$ by Lemma 5.2.8 since $Y^{\rm b}\in\mathcal{S}_T^p(\mathbb{F},\mathbb{Q})$ and $Y^{\rm d}(\tau)\in\mathcal{S}_T^p(\mathbb{G}^\tau,\mathbb{P})$. The semimartingale is therefore special by [Pro04, Theorem 33 of chapter 3].

$D = N^{\mathbb{G}} + \Lambda^{\mathbb{G}}$ with $\Lambda^{\mathbb{G}}_t = \int_0^t \lambda_s 1_{\{s\le\tau\}}ds, t\in[0,T]$, it follows from Theorem 2 in [CMS80] that for every $t\in[0,T]$

$$\int_0^t U_s dD_s \;=\; \int_0^t U_s dN^{\mathbb{G}}_s + \int_0^t U_s d\Lambda^{\mathbb{G}}_s = \int_0^t U_s dN^{\mathbb{G}}_s + \int_0^t U_s\lambda_s ds. \tag{5.43}$$

We deduce that for $t\in[0,T]$

$$\int_0^t (Y^{\mathrm{d}}_s(s) - Y^{\mathrm{b}}_{s-})dD_s - (1-D_{s-})(Y^{\mathrm{d}}_s(s) - Y^{\mathrm{b}}_{s-})\lambda_s ds = \int_0^t U_s dD_s - \int_0^t U_s d\Lambda^{\mathbb{G}}_s = \int_0^t U_s dN^{\mathbb{G}}_s.$$

Using the expressions of F, Y, Z and U, we have for $t\in[0,T]$

$$\begin{aligned}\int_0^t (1-D_{s-})F^{\mathrm{b}}(s, Y^{\mathrm{b}}_{s-}, Z^{\mathrm{b}}_s, Y^{\mathrm{d}}_s(s) - Y^{\mathrm{b}}_{s-})ds - D_{s-}F^{\mathrm{d}}(s, Y^{\mathrm{d}}_{s-}(\tau), Z^{\mathrm{d}}_s(\tau))ds&\\ = \int_0^t F(s, Y_{s-}, Z_s, U_s)ds.&\end{aligned}$$

Hence it follows from properties of integration that for $t\in[0,T]$

$$\begin{aligned}\int_0^t (1-D_{s-})dK^{\mathrm{b}}_s + D_{s-}dK^{\mathrm{d}}_s(\tau) &= K^{\mathrm{b}}_{t\wedge\tau} + (K^{\mathrm{d}}_t(\tau) - K^{\mathrm{d}}_\tau(\tau))1_{\{t\ge\tau\}} = K_t,\\ \int_0^t (1-D_{s-})dM^{\mathrm{b}}_s + D_{s-}dM^{\mathrm{d}}_{s-}(\tau) &= M^{\mathrm{b}}_{t\wedge\tau} + (M^{\mathrm{d}}_t(\tau) - M^{\mathrm{d}}_\tau(\tau))1_{\{t\ge\tau\}} = M_t.\end{aligned}$$

Using the definition Z and Lemma 5.3.8, we have

$$\int_0^t (1-D_{s-})Z^{\mathrm{b}}_s dB^{\mathbb{F}}_s + \int_0^t D_{s-}Z^{\mathrm{d}}_s(\tau)dB^{\mathbb{G}^\tau}_s \;=\; \int_0^t Z_s dB^{\mathbb{G}}_s,\ t\in[0,T].$$

Inserting the above equalities into the equation for Y, we see that (Y, Z, U, M, K) satisfies

$$dY_t \;=\; -F(t, Y_{t-}, Z_t, U_t)dt - dK_t + Z_t dB^{\mathbb{G}}_t + U_t dN^{\mathbb{G}}_t + dM_t,\ t\in[0,T].$$

Step 3. We show that the Skorohod condition $\int_0^T (Y_{s-} - S_{s-})dK_s = 0$ holds. Since the quadruplet $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, K^{\mathrm{b}}, M^{\mathrm{b}})$ is a solution to the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_{Y^{\mathrm{d}}_\cdot(\cdot)}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$, we have the inequalities $\int_0^T (Y^{\mathrm{b}}_{s-} - S^{\mathrm{b}}_{s-})dK^{\mathrm{b}}_s = 0$ and $Y^{\mathrm{b}} \ge S^{\mathrm{b}}$. Thus $\int_u^v (Y^{\mathrm{b}}_{s-} - S^{\mathrm{b}}_{s-})dK^{\mathrm{b}}_s = 0$ for all $[u,v]\subseteq[0,T]$. Hence

$$\begin{aligned}\int_0^{T\wedge\tau} (Y_{s-} - S_{s-})dK_s &= \int_0^T (Y_{s-} - S_{s-})dK_s 1_{\{T<\tau\}} + \int_0^\tau (Y_{s-} - S_{s-})dK_s 1_{\{T\ge\tau\}}\\ &= \int_0^T (Y^{\mathrm{b}}_{s-} - S^{\mathrm{b}}_{s-})dK^{\mathrm{b}}_s 1_{\{T<\tau\}} + \int_0^\tau (Y^{\mathrm{b}}_{s-} - S^{\mathrm{b}}_{s-})dK^{\mathrm{b}}_s 1_{\{T\ge\tau\}} = 0.\end{aligned}$$

By the hypothesis on $(Y^{\mathrm{d}}(\tau), S^{\mathrm{d}}(\tau)D, \xi^{\mathrm{d}}(\tau)D)$, we have $\int_0^T (Y^{\mathrm{d}}_{s-}(\tau) - S^{\mathrm{d}}_{s-}(\tau)D_{s-})dK^{\mathrm{d}}_s(\tau) = 0$. We infer from the splitting formula (5.28) of (Y, S, K) that

$$\begin{aligned}\int_{T\wedge\tau}^T (Y_{s-} - S_{s-})dK_s &= \int_\tau^T (Y_{s-} - S_{s-})dK_s 1_{\{T>\tau\}}\\ &= \int_\tau^T (Y^{\mathrm{d}}_{s-}(\tau) - S^{\mathrm{d}}_{s-}(\tau)D_{s-})dK^{\mathrm{d}}_s(\tau)1_{\{T>\tau\}} = 0.\end{aligned}$$

The Skorohod condition is thus satisfied. Clearly we have $Y\ge S$ and $Y_T = \xi$.
Steps 1, 2,3 show that (Y, Z, U, M, K) is an $\mathcal{S}^p$-solution to the RBSDE(F, S, ξ). $\square$

Remark 5.3.9. *We keep the notation of Theorem 5.3.5. Note that $Y^d(\tau)$ being a $\mathbb{G}^\tau$-optional process, Proposition 5.2.2 implies that $Y_\cdot^d(\cdot)$ is $\mathcal{O}(\mathbb{F})$-measurable. The predictability assumption on $Y_\cdot^d(\cdot)$ in Theorem 5.3.5 is necessary for the process U to be $\mathcal{P}(\mathbb{F})$-measurable so that we can interpret $\int_0^\cdot U_s dD_s$ as a stochastic integral and use the canonical decomposition (5.43). Some cases in which $Y_\cdot^d(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable are:*

- *The filtration $\mathbb{F}$ is continuous: Then $\mathcal{O}(\mathbb{F}) = \mathcal{P}(\mathbb{F})$ and thus $Y_\cdot^d(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable.*
- *$F^d = 0$ and there exists an $\mathbb{F}$-predictable process ζ such that $S_t^d(\tau)1_{\{t\geq\tau\}} = \zeta_\tau 1_{\{t\geq\tau\}}$, $t \in [0,T]$ and $\xi^d(\tau) = \zeta_\tau$. Then using the Snell envelope representation given by (5.24) and the properties of* ess sup*, we obtain that $Y_t^d(\tau)1_{\{t\geq\tau\}} = \zeta_\tau 1_{\{t\geq\tau\}}, t \in [0,T]$. In particular $Y_\tau^d(\tau)1_{\{T\geq\tau\}} = \zeta_\tau 1_{\{T\geq\tau\}}$ and Lemma 4.2.16 entails that $Y^d(\cdot) = \zeta, dt\otimes\mathbb{P}$-a.e.. We can w.l.o.g. choose $Y_\cdot^d(\cdot) = \zeta$ and thus $Y_\cdot^d(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable as ζ is.*

Remark 5.3.10. *Theorem 5.3.5 remains true if we consider $\big(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau)\big)$ as the solution to the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^d, S^d(\tau), \xi^d(\tau))$. This is due to the fact that the data $(F^d, S^d(\tau), \xi^d(\tau))$ and $(F^d D_-, S^d(\tau)D, \xi^d(\tau)D_T)$ are indistinguishable after τ. Moreover, **Step 2** and **Step 3** of the proof rely only on the dynamical description and value of the solution $(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau))$ of the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^d D_-, S^d(\tau)D, \xi^d(\tau)D_T)$ after τ. In Section 5.4 where we work with F^d having quadratic growth in z, the continuity of the obstacle is needed for the existence of a solution to the RBSDE satisfied by the post-default value of (Y, Z, U, M, K). For this reason, we will consider the data $(F^d, S^d(\tau), \xi^d(\tau))$ and impose continuity on $S^d(\tau)$.*

Remark 5.3.11. *Note that Theorem 5.3.5 can be applied to construct a solution $(Y, Z, U, M) \in \mathcal{S}_T^p(\mathbb{G},\mathbb{P}) \times \mathcal{H}_T^{2,n}(\mathbb{G},\mathbb{P}) \times \mathcal{L}_T^p(\mathbb{G},\mathbb{P}) \times \mathcal{M}_T^p(\mathbb{G},\mathbb{P})$ to the BSDE(F,ξ), i.e.*

$$Y_t = \xi + \int_t^T F(s, Y_{s-}, Z_s, U_s)ds - \int_t^T Z_s dB_s^{\mathbb{G}} - \int_t^T U_s dN_s^{\mathbb{G}} - \int_t^T dM_s,\ \ t \in [0,T]. \tag{5.44}$$

*Indeed, it follows from the arguments of **Step 1** and **Step 2** that it suffices to construct a solution $(Y^d(\tau), Z^d(\tau), M^d(\tau)) \in \mathcal{S}_T^p(\mathbb{G}^\tau) \times \mathcal{H}_T^{2,n}(\mathbb{G}^\tau,\mathbb{P}) \times \mathcal{M}_{\mathbb{G}^\tau}^p(B^{\mathbb{G}^\tau},\mathbb{P})$ and $(Y^b, Z^b, M^b) \in \mathcal{S}_T^p(\mathbb{F},\widehat{\mathbb{Q}}) \times \mathcal{H}_T^{2,n}(\mathbb{F},\widehat{\mathbb{Q}}) \times \mathcal{M}_{\mathbb{F}}^p(B^{\mathbb{F}},\widehat{\mathbb{Q}})$ such that for $t \in [0,T]$*

$$\begin{aligned} Y_t^d(\tau) &= \xi^d(\tau)D_T + \int_t^T F^d\big(s, Y_{s-}^d(\tau), Z_s^d(\tau)\big)D_{s-}ds - \int_t^T Z_s^d(\tau)dB_s^{\mathbb{G}^\tau} - \int_t^T dM_s^d(\tau), \\ Y_t^b &= \xi^b + \int_t^T \big[F^b\big(s, Y_{s-}^b, Z_s^b, Y_s^d(s) - Y_{s-}^b\big) + \lambda_s\big(Y_s^d(s) - Y_{s-}^b\big)\big]ds - \int_t^T Z_s^b dB_s^{\mathbb{F}} - \int_t^T dM_s^b, \end{aligned}$$

and then define (Y, Z, U, M) as in (5.28). In the context of BSDEs a similar pasting procedure has been developed in [ABSEL10, KL12] for $\mathbb{F}$ being the completion of the filtration generated by the Brownian motion B and for F having quadratic growth in z (This will be discussed in Section 5.4).

The following assumption collects the conditions under which we give the existence of an $\mathcal{S}^p$-solution to (5.13) for some $p > 1$. The case of a bounded solution is discussed in Section 5.4.

Assumption 5.3.12. *Let $p > 1$.*

(H1) $S \in \mathcal{S}_T^p(\mathbb{G},\mathbb{P})$ *and* $\mathbb{E}\left[|\xi|^p + \left(\int_0^T |F(s,0,0,0)|^2 ds\right)^{\frac{p}{2}}\right] < +\infty.$

(H2) There exists a constant δ_L such that for every $y_1, y_2, u_1, u_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^n$ and $t \in [0,T]$ we have

$$|F(t,\cdot,y_1,z_1,u_1) - F(t,\cdot,y_2,z_2,u_2)| \leq \delta_L\left(|y_1 - y_2| + ||z_1 - z_2|| + \sqrt{\lambda_t}1_{\{t\leq\tau\}}|u_1 - u_2|\right).$$

The condition (H2) is the familiar Lipschitz assumption on the driver F which together with (H1) guarantees the existence of a solution (Y, Z, U, M, K) to (5.13). Two cases are to be considered. The case $p \geq 2$ and $\mathbb{F}$ is the completion of the filtration generated by the Brownian motion B. Then the pair $(B^{\mathbb{G}}, N^{\mathbb{G}})$ admits the *predictable representation property* in the filtration $\mathbb{G}$ (see [CJZ13, Proposition 5.5.]). Thus we are in the classical situation of RBSDEs with jumps for which we can apply the contraction principle or the penalization method as in [HO16, QS14, Ess08]. The other case deals with $p \in (1,2)$ and the filtration $\mathbb{F}$ being not quasi-left continuous. The first difficulty is due to the fact that for $p \in (1,2)$, one requires a more general Itô's formula (see [KP15, Lemma 7]) to obtain estimates of the solutions which permit the application of the contraction principle. The second difficulty stems from the fact that for $\mathbb{F}$ not quasi-left continuous, $\mathbb{G}$ is not quasi-left continuous and the processes M and K in the solution (Y, Z, U, M, K) can jump at the same time. Hence the process $[M, K]$ does not vanish and has to be controlled appropriately when deriving estimates of solutions. These difficulties have been overcome recently in [BPTZ15] where the existence of an $\mathcal{S}^p$-solution to (5.13) has been established. Note however that in [BPTZ15] there is no pure jump driven martingale and the driver depends only on y, z. The presence of the jump martingale $N^{\mathbb{G}}$ and the additional dependence of F on u induce a new difficulty since we have to take care of the compensator terms resulting from Itô's formula.

Theorem 5.3.5 shows that existence of solutions for RBSDE (5.17) and RBSDE (5.19) ensures the existence of a solution to the RBSDE(F, S, ξ). In the sequel we employ Theorem 5.3.5 to show the existence of an $\mathcal{S}^p$-solution to the RBSDE(F, S, ξ).

Theorem 5.3.13. *Suppose that Assumption 5.3.12 (H1) and (H2) hold. Let $p > 1$. Then the RBSDE (5.17) admits a unique $\mathcal{S}^p$-solution $(Y^d(\tau), Z^d(\tau), M^d(\tau), K^d(\tau))$. We consider the assertions:*

i) $p \geq 2$ and $Y^d_\cdot(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable.

ii) $p \in (1,2)$ and $\sqrt{\lambda} Y^d_\cdot(\cdot) \in \mathcal{H}^{p,1}_T(\mathbb{F}, \widehat{\mathbb{Q}})$.

If i) or ii) holds, then RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^b_{Y^d_\cdot(\cdot)}, S^b, \xi^b)$ admits a unique $\mathcal{S}^p$-solution (Y^b, Z^b, M^b, K^b). Thus the RBSDE(F, S, ξ) admits an $\mathcal{S}^p$-solution (Y, Z, U, M, K) given by (5.28).

Proof. We begin with the first statement. Using the formulas of ξ, S and F given respectively by (5.10), (5.11) and (5.12) together with (H1) and (H2), we have

$$|F^{\mathrm{d}}(t, y, z)D_{t-} - F^{\mathrm{d}}(t, y', z')D_{t-}| \quad \leq C_y(|y - y'| + ||z - z'||), \ y, y' \in \mathbb{R}, z, z' \in \mathbb{R}^n.$$

Moreover, $S^{\mathrm{d}}(\tau)D \in \mathcal{S}^p_T(\mathbb{G}^\tau, \mathbb{P})$ and

$$\mathbb{E}\left[(\xi^{\mathrm{d}}(\tau)D_T)^p + \left(\int_0^T |F^{\mathrm{d}}(s, 0, 0)D_{s-}|^2 ds\right)^{\frac{p}{2}}\right] \quad \leq \mathbb{E}\left[\xi^p + \left(\int_0^T |F(s, 0, 0, 0)|^2 ds\right)^{\frac{p}{2}}\right] < +\infty.$$

Due the conditions above, the data $(F^{\mathrm{d}} D_-, S^{\mathrm{d}}(\tau)D, \xi^{\mathrm{d}}(\tau)D_T)$ satisfy the assumptions of Theorem 3.1 in [BPTZ15]. Hence there exists a unique quadruplet $(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), M^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau)) \in \mathcal{S}_{\mathbf{sol}}(\mathbb{G}^\tau, \mathbb{P})$ such that

$$\begin{cases} dY^{\mathrm{d}}_t(\tau) & = & -F^{\mathrm{d}}(t, Y^{\mathrm{d}}_{t-}(\tau), Z^{\mathrm{d}}_t(\tau))D_{t-}dt - dK^{\mathrm{d}}_t(\tau) + dB^{\mathbb{G}^\tau}_t + dM^{\mathrm{d}}_t(\tau), \ t \in [0, T], \\ Y^{\mathrm{d}}_t(\tau) & \geq & S^{\mathrm{d}}_t(\tau)D_t, \ t \in [0, T], \\ Y^{\mathrm{d}}_T(\tau) & = & \xi^{\mathrm{d}}(\tau)D_T \text{ and } \int_0^T \left(Y^{\mathrm{d}}_{t-}(\tau) - S^{\mathrm{d}}_{t-}(\tau)D_{t-}\right) dK^{\mathrm{d}}_t(\tau) = 0. \end{cases}$$

This proves the first statement.

We now suppose that $Y^{\mathrm{d}}_\cdot(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable. Recall that for $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n$

$$F^{\mathrm{b}}_{Y^{\mathrm{d}}_\cdot(\cdot)}(s, \cdot, y, z) \quad = \quad F^{\mathrm{b}}(s, \cdot, y, z, Y^{\mathrm{d}}_\cdot(\cdot) - y) + \lambda_t(Y^{\mathrm{d}}_\cdot(\cdot) - y). \tag{5.45}$$

Since $|\lambda|_\infty < +\infty$, and F is uniformly Lipschitz in (y,z,u) by (H2), $F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)}$ is uniformly Lipschitz in (y,z). Clearly $F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)}(t,\cdot,0,0) = F^{\mathrm{b}}(t,\cdot,0,0,Y^{\mathrm{d}}_t(t)) + \lambda_t Y^{\mathrm{d}}_t(t), t\in[0,T]$. Using (H2), we get

$$|F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)}(t,\cdot,0,0)| \quad \leq \quad |F^{\mathrm{b}}(t,0,0,0)| + \sqrt{\lambda_t}|Y^{\mathrm{d}}_t(t)| + \lambda_t|Y^{\mathrm{d}}_t(t)|,\ t\in[0,T]. \tag{5.46}$$

As $|\lambda|_\infty < +\infty$, there exists a constant C_λ such that

$$|F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)}(t,0,0,0)|^2 \leq C_\lambda\left(|F^{\mathrm{b}}(t,0,0,0)|^2 + \lambda|Y^{\mathrm{d}}_t(t)|^2\right),\ t\in[0,T]. \tag{5.47}$$

By (H1), $F(\cdot,0,0,0)\in\mathcal{H}^{p,1}_T(\mathbb{G},\mathbb{P})$ and Lemma 5.2.8 guarantee that $F^{\mathrm{b}}(\cdot,0,0,0)\in\mathcal{H}^{p,1}_T(\mathbb{F},\widehat{\mathbb{Q}})$. Now if $p\geq 2$, then $\sqrt{\lambda}Y^{\mathrm{d}}_{\cdot}(\cdot)\in\mathcal{H}^{p,1}_T(\mathbb{F},\widehat{\mathbb{Q}})$ by Lemma 5.2.8 since $Y^{\mathrm{d}}(\tau)\in\mathcal{S}^p(\mathbb{G}^\tau,\mathbb{P})$. Hence (5.47) implies that $F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)}(\cdot,0,0,0)\in\mathcal{H}^{p,1}_T(\mathbb{F},\widehat{\mathbb{Q}})$ and the existence of a unique $\mathcal{S}^p$-solution to $\mathrm{RBSDE}_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)},S^{\mathrm{b}},\xi^{\mathrm{b}})$ follows from Theorem 3.1 in [BPTZ15]. We infer from Theorem 5.3.5 that (Y,Z,U,M,K) defined by (5.28) is an $\mathcal{S}^p$-solution to the RBSDE(F,S,ξ).
For $p\in(1,2)$ and $\sqrt{\lambda}Y^{\mathrm{d}}_{\cdot}(\cdot)\in\mathcal{H}^{p,1}_T(\mathbb{F},\widehat{\mathbb{Q}})$, again (5.47) ensures that $F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)}(\cdot,0,0,0)\in\mathcal{H}^{p,1}_T(\mathbb{F},\widehat{\mathbb{Q}})$ and similar arguments as for the case $p\geq 2$ yield the existence of an $\mathcal{S}^p$-solution to the $\mathrm{RBSDE}_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)},S^{\mathrm{b}},\xi^{\mathrm{b}})$ and thus to the RBSDE(F,S,ξ). □

5.4 Existence of solutions for drivers with quadratic growth

We illustrate in this section the use of the decomposition approach given by Theorem 5.3.5 to show the existence of bounded solutions to the RBSDE(F,S,ξ) for the class of drivers having quadratic growth in z and exponential growth in u. We restrict ourselves to the following type of filtrations $\mathbb{F}$:

Assumption 5.4.1. *$\mathbb{F}$ is the completion of the filtration generated by the Brownian motion B.*

The following proposition from [CJZ13] discusses some consequences of Assumption 5.4.1 for the *predictable martingale representation property* in the filtrations $\mathbb{G}$ and $\mathbb{G}^\tau$.

Proposition 5.4.2. *Under Assumption 5.4.1, we have:*

i) For every $(\mathbb{G},\mathbb{P})$-locally square integrable martingale M, there exist $Z\in L_{\mathbb{G}}(B^{\mathbb{G}})$ and $U\in L_{\mathbb{G}}(N^{\mathbb{G}})$ such that

$$M \quad = \quad M_0 + \int_0^{\cdot} ZdB^{\mathbb{G}} + \int_0^{\cdot} UdN^{\mathbb{G}}. \tag{5.48}$$

ii) For every $(\mathbb{G}^\tau,\mathbb{P})$-locally square integrable martingale $M^d(\tau)$, there exists a predictable process $Z^d(\tau)\in L_{\mathbb{G}^\tau}(B^{\mathbb{G}^\tau})$ such that

$$M^d(\tau) \quad = \quad M^d_0(\tau) + \int_0^{\cdot} Z^d(\tau)dB^{\mathbb{G}^\tau}. \tag{5.49}$$

Proof. See [CJZ13, Proposition 5.5]. □

In view of Part i) of Proposition 5.4.2, every $(\mathbb{G},\mathbb{P})$-locally square integrable martingale M belonging to the space $\mathcal{M}^{loc}_{\mathbb{G}}(B^{\mathbb{G}},\mathbb{P})\cap\mathcal{M}^{loc}_{\mathbb{G}}(N^{\mathbb{G}},\mathbb{P})$ is the null martingale. Now note that if (Y,Z,U,K) is a quadruplet of $\mathbb{G}$-adapted processes satisfying

$$\begin{cases} Y_t = \xi + \int_t^T F(s,Y_{s-},Z_s,U_s)ds - \int_t^T Z_s dB^{\mathbb{G}}_s - \int_t^T U_s dN^{\mathbb{G}}_s + K_T - K_t,\ t\in[0,T],\\ Y_t \geq S_t,\ t\in[0,T],\\ \int_0^T (Y_{s-}-S_{s-})dK_s = 0, \end{cases} \tag{5.50}$$

then $(Y, Z, U, 0, K)$ is a solution to the RBSDE(F, S, ξ). We can therefore refer to the system of equations (5.50) as the RBSDE(F, S, ξ) and a quadruplet of processes (Y, Z, U, K) satisfying (5.50) as a solution to the RBSDE(F, S, ξ).
Regarding the terminal value ξ and the obstacle process S we require the following conditions on their pre-default and post-default values:

Assumption 5.4.3. *Let ξ and S be given respectively by (5.10) and (5.11). We suppose that*

(H1') The processes S^b and $S^d(\tau)$ have continuous paths. Moreover, $S^b \in \mathcal{S}_T^\infty(\mathbb{F})$ and $S^d(\tau) \in \mathcal{S}_T^\infty(\mathbb{G}^\tau)$.

(H1") $\xi^b \in L^\infty(\mathbb{F})$ and $\xi^d(\tau) \in L^\infty(\mathbb{G}^\tau)$.

Before being more precise on the assumptions for the driver, let us introduce the condition $(\mathbf{A}_\gamma)$ which plays a useful role in establishing the existence and uniqueness of solutions for the RBSDE(F, S, ξ).

Definition 5.4.4. *We say that a $\mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^{n+2}) - \mathcal{B}(\mathbb{R})$-measurable function $g : [0, T] \times \Omega \times \mathbb{R}^{n+2} \to \mathbb{R}$, satisfies the condition $(\mathbf{A}_\gamma)$ if for every $L > 0$, there exists a $\mathcal{P}(\mathbb{H}) \otimes \mathcal{B}(\mathbb{R}^{n+3}) \otimes \mathcal{B}([-L, L]^2) - \mathcal{B}(\mathbb{R})$-measurable function $\Pi : [0, T] \times \Omega \times \mathbb{R}^{n+3} \times [-L, L]^2 \to \mathbb{R}$, such that for every $(y, z, u, u') \in \mathbb{R}^{n+1} \times [-L, L]^2$ and for $dt \otimes \mathbb{P}$-a.e. $(t, \omega) \in [0, T] \times \Omega$ we have*

$$g(t, \omega, y, z, u) - g(t, \omega, y, z, u') \quad \leq \quad \lambda_t(\omega)\Pi^{y,z,u,u'}(t, \omega)(u - u').$$

Moreover there exists a constant δ_L such that $-1 \leq \Pi^{y,z,u,u'}(t, \omega) \leq \delta_L, dt \otimes \mathbb{P}$-a.e. $(t, \omega) \in [0, T] \times \Omega$.

Remark 5.4.5. *The condition $(\mathbf{A}_\gamma)$ appears in [QS13, KP15] and [QS14] respectively in the context of BSDEs and RBSDEs with jumps driven by a Poisson process. It is imposed to at least one driver when establishing the comparison principle for BSDEs or RBSDEs with jumps.*
Observe that the condition $(\mathbf{A}_\gamma)$ on g implies that g is locally Lipschitz in u. Note that we require only $-1 \leq \Pi^{y,z,u,u'}(t, \omega)$ rather than the restrictive assumption $-1 + \delta \leq \Pi^{y,z,u,u'}(t, \omega)$ for some $\delta > 0$. The latter is the usual $(\mathbf{A}_\gamma)$ condition introduced in [Roy06] in the context of BSDEs with jumps for the purpose of deriving a comparison principle.

We will consider the following set of conditions on F^{d} and F^{b}:

Assumption 5.4.6. *There exist $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) - \mathcal{B}(\mathbb{R})$-measurable functions $\Sigma, J : [0, T] \times \Omega \times \mathbb{R} \to \mathbb{R}$, and positive constants C_1, C_2, C_y, C_z such that:*

(Q1) For all $(t, \omega) \in [0, T] \times \Omega$, $F^b(t, \omega, \cdot, \cdot, \cdot)$ and $F^d(t, \omega, \cdot, \cdot)$ are continuous. Moreover for all $(y, z, u) \in \mathbb{R}^{n+2}$ we have $dt \otimes \mathbb{P}$-a.e.

$$\begin{aligned} -C_2 - C_y|y| - C_z||z||^2 - \lambda\Sigma(u) \leq & F^b(\cdot, y, z, u) \leq C_1 + C_y|y| + C_z||z||^2 + \lambda J(u) \\ -C_2 - C_y|y| - C_z||z||^2 \leq & F^d(\cdot, y, z) \leq C_1 + C_y|y| + C_z||z||^2. \end{aligned}$$

(Q2) For every $L > 0$, there exists a constant $\delta_{\Sigma_L} > 0$ such that for every $u, u' \in [-L, L]$

$$|\Sigma_t(u) - \Sigma_t(u')| \leq \delta_{\Sigma_L}|u - u'|, \forall\ t \ \in [0, T]\ \mathbb{P}\text{-a.s.}$$

(Q3) The function λJ satisfies the condition $(\mathbf{A}_\gamma)$.

(Q4) The processes $J(0), \Sigma(0)$ and $F^d(\cdot, 0, 0)$ are bounded.

Using (Q1) and (Q2) we have for $(t, y, z, u) \in [0, T] \times \mathbb{R}^{n+2}$,

$$-C_2 - C_y|y| - C_z||z||^2 + \lambda_t \Sigma_t(u) 1_{\{t\le\tau\}} \le F(t, y, z, u) \le C_1 + C_y|y| + C_z||z||^2 + \lambda_t J_t(u) 1_{\{t\le\tau\}}. \quad (5.51)$$

Remark 5.4.7. *It follows from the bound (5.51) that F has quadratic growth in z. A particular choice of Σ and J appearing in the literature of BSDEs [ABSEL10, Ngo10, JMN12, JMPR15] are for example*

$$\Sigma(u) = \frac{1}{C_z}\left(e^{-C_z u} + C_z u - 1\right) \text{ and } J(u) = \frac{1}{C_z}\left(e^{C_z u} - C_z u - 1\right), \quad u \in \mathbb{R}. \quad (5.52)$$

The driver F is therefore allowed to have exponential growth in the jump variable u. In the particular case of where Σ and J are given by (5.52), solutions to BSDEs with drivers satisfying (5.51) belong to the class of quadratic exponential semimartingales *introduced in [Ngo10, EKMN16].*

Our goal in the sequel is to show the existence of a bounded solution (Y, Z, U, K) to the RBSDE (5.50) under Assumptions 5.4.3 and 5.4.6. First we show that for a solution (Y, Z, U, K) of (5.50), the boundedness of Y confers nice integrability properties to the triplet (Z, U, K). We begin with the following simple lemma regarding the pointwise boundedness of U.

Lemma 5.4.8. *Let (Y, Z, U, K) be a bounded solution to (5.50). Then $|U| \le 2||Y||_\infty$, $dt\otimes\mathbb{P}$-a.e..*

Proof. Following the proof of Proposition 5.3.4, we can assume w.l.o.g. that there exists a $\mathcal{P}(\mathbb{F})$-measurable process $\widetilde{U}$ such that $U_t = \widetilde{U}_t 1_{\{t\le\tau\}}$, $t \in [0, T]$. Since K is $\mathcal{P}(\mathbb{G})$-measurable and τ is totally inaccessible the jump size of Y at τ is given by $\widetilde{U}_\tau$. Thus $|\widetilde{U}_\tau 1_{\{\tau\le T\}}| = |\Delta Y_\tau 1_{\{\tau\le T\}}| \le 2||Y||_\infty$. By Lemma 4.2.16, $|\widetilde{U}| \le 2||Y||_\infty$, $dt \otimes \mathbb{P}$-a.e.. □

The following proposition shows that the boundedness of Y confers to its martingale part a BMO property and that the increasing process K belongs to the space $\mathcal{IS}_{BMO}(\mathbb{G}, \mathbb{P})$. It has been recently established in [Lio14] in the case without jumps.

Proposition 5.4.9. *Suppose that Assumptions 5.4.3 and 5.4.6 hold. Let (Y, Z, U, K) be a bounded solution to (5.50). Then $(Z, U, K) \in \mathcal{H}^{2,n}_{BMO}(\mathbb{G}, \mathbb{P}) \times \mathcal{L}^2_{BMO}(\mathbb{G}, \mathbb{P}) \times \mathcal{IS}_{BMO}(\mathbb{G}, \mathbb{P})$.*

Proof. Assume that Y is bounded. As $U \cdot N^{\mathbb{G}}$ has quadratic variation $U^2_\tau D$, we deduce from Lemma 5.4.8 that $U \in \mathcal{L}^2_{BMO}(\mathbb{G}, \mathbb{P})$. We now show that $(Z, K) \in \mathcal{H}^{2,n}_{BMO}(\mathbb{G}, \mathbb{P}) \times \mathcal{IS}_{BMO}(\mathbb{G}, \mathbb{P})$. To achieve this, we follow the same line of arguments as in [Lio14]. Let $\mu \in \mathbb{R}$ to be chosen later. Let $\sigma \in \mathcal{T}_T(\mathbb{G})$. By Itô's formula and the dynamical description (5.50) of Y, we have

$$\begin{aligned} e^{\mu Y_\sigma} + \frac{1}{2}\mu^2 \int_\sigma^T e^{\mu Y_{s-}} ||Z_s||^2 ds - \mu \int_\sigma^T e^{\mu Y_{s-}} dK_s + \sum_{\sigma<s\le T} \left\{ e^{\mu Y_s} - e^{\mu Y_{s-}} - \mu e^{\mu Y_{s-}} \Delta Y_s \right\} \\ = e^{\mu Y_T} - \mu \int_\sigma^T e^{\mu Y_{s-}} \left(-F(s, Y_{s-}, Z_s, U_s) ds + U_s dN^{\mathbb{G}}_s + Z_s dB^{\mathbb{G}}_s \right). \end{aligned}$$

Clearly, $\sum_{\sigma<s\le T} \left\{ e^{\mu Y_s} - e^{\mu Y_{s-}} - \mu e^{\mu Y_{s-}} \Delta Y_s \right\} = \sum_{\sigma<s\le T} \left\{ e^{\mu Y_{s-}} \left(e^{\mu \Delta Y_s} - 1 - \mu \Delta Y_s \right) \right\} \ge 0$ due to the positivity of the function $\mathbb{R} \ni x \mapsto e^x - 1 - x$. We infer from the above equality that

$$\begin{aligned} \frac{1}{2}\mu^2 \int_\sigma^T e^{\mu Y_{s-}} ||Z_s||^2 ds - \mu \int_\sigma^T e^{\mu Y_{s-}} dK_s \\ \le e^{\mu Y_T} - \mu \int_\sigma^T e^{\mu Y_{s-}} \left(-F(s, Y_{s-}, Z_s, U_s) ds + U_s dN^{\mathbb{G}}_s + Z_s dB^{\mathbb{G}}_s \right). \end{aligned}$$

The processes Y and U being bounded, the local martingales $\int_0^\cdot e^{\mu Y_{s-}} U_s dN^{\mathbb{G}}_s$ and $\int_0^\cdot e^{\mu Y_{s-}} Z_s dB^{\mathbb{G}}_s$ are true martingales. Taking the conditional expectations in the above inequality w.r.t. $\mathcal{G}_\sigma$ leads to

$$\begin{aligned}&\frac{1}{2}\mu^2\mathbb{E}\left[\int_\sigma^T e^{\mu Y_{s-}}||Z_s||^2 ds|\mathcal{G}_\sigma\right] - \mu\mathbb{E}\left[\int_\sigma^T e^{\mu Y_{s-}} dK_s|\mathcal{G}_\sigma\right]\\ &\le \mathbb{E}\left[e^{\mu Y_T}|\mathcal{G}_\sigma\right] + \mu\mathbb{E}\left[\int_\sigma^T e^{\mu Y_{s-}} F(s, Y_{s-}, Z_s, U_s) ds|\mathcal{G}_\sigma\right]. \end{aligned} \tag{5.53}$$

Using the bounds of F given by (5.51), one obtains

$$|F(s, Y_{s-}, Z_s, U_s)| \quad \le \quad C_1 + C_2 + C_y|Y_{s-}| + C_z||Z_s||^2 + |\lambda_s\Sigma_s(U_s)| + |\lambda_s J_s(U_s)|, \; s \in [0, T].$$

As U is bounded, the properties (Q2) and (Q3) imply that there exists $C > 0$ such that

$$|\lambda\Sigma(U)| + |\lambda J(U)| < C. \tag{5.54}$$

Taking into account the latter estimate of $F(\cdot, Y_-, Z, U)$, we obtain from (5.53) after rearranging terms that

$$\left(\frac{1}{2}\mu^2 - |\mu|C_z\right)\mathbb{E}\left[\int_\sigma^T e^{\mu_s Y_{s-}}||Z_s||^2 ds|\mathcal{G}_\sigma\right] - \mu\mathbb{E}\left[\int_\sigma^T e^{\mu_s Y_{s-}} dK_s|\mathcal{G}_\sigma\right] \quad \le \quad e^{|\mu|||Y||_\infty}C_\infty,$$

with $C_\infty = (1 + |\mu|T(C + C_1 + C_2 + C_y||Y||_\infty))$. We choose $\mu = -5C_z$. Then $\frac{1}{2}\mu^2 - C_z|\mu| = \frac{15}{2}C_z^2$. Using the inequality $e^{-\mu||Y||_\infty} \le e^{-\mu Y_{s-}} \le e^{|\mu|||Y||_\infty}$ we obtain

$$\frac{15}{2}C_z^2\mathbb{E}\left[\int_\sigma^T ||Z_s||^2 ds|\mathcal{G}_\sigma\right] + 5C_z\mathbb{E}\left[K_T - K_\sigma|\mathcal{G}_\sigma\right] \quad \le \quad e^{10C_z||Y||_\infty}C_\infty.$$

Since σ is arbitrary, we deduce that $(Z, U, K) \in \mathcal{H}^{2,n}_{BMO}(\mathbb{G}, \mathbb{P}) \times \mathcal{L}^2_{BMO}(\mathbb{G}, \mathbb{P}) \times \mathcal{IS}_{BMO}(\mathbb{G}, \mathbb{P})$. □

Proposition 5.4.9 shows that $\mathcal{S}^\infty_T(\mathbb{G}) \times \mathcal{H}^{2,n}_{BMO}(\mathbb{G}, \mathbb{P}) \times \mathcal{L}^2_{BMO}(\mathbb{G}, \mathbb{P}) \times \mathcal{IS}_{BMO}(\mathbb{G}, \mathbb{P})$ is the natural normed space for a bounded solution. Taking this into account, we will use techniques from BSDEs [AIDR07, ABSEL10] to provide *a priori* estimates in Section 5.4.1 ensuring the uniqueness of a bounded solution, see Corollary 5.4.16.

5.4.1 A priori estimates of bounded solutions

In this section, we present an analysis of the norm of bounded solutions to RBSDEs (5.50) w.r.t. their input data. The a priori estimates we give here follow the same guideline as the *a priori* estimates for BSDEs of quadratic growth with a single jump which appear in [ABSEL10]. However additional attention is required for the treatment of the increasing process K of a solution (Y, Z, U, K). Our analysis will be carried out for drivers with pre-default and post-default values satisfying the following assumption:

Assumption 5.4.10. *For a driver F with pre-default and post-default values F^b and F^d, we suppose that F^b and F^d have the following properties:*

(L1) There exists constants C_y and C_z such that $\forall\ y, y', u, z, z' \in \mathbb{R}$

$$\begin{aligned}|F^d(\cdot, y, z) - F^d(\cdot, y', z)| + |F^b(\cdot, y, z, u) - F^b(\cdot, y', z, u)| &\le C_y|y - y'|\\ |F^d(\cdot, y, z) - F^d(\cdot, y, z') &\le C_z(1 + ||z|| + ||z'||)||z - z'||,\\ |F^b(\cdot, y, z, u) - F^b(\cdot, y, z', u)| &\le C_z(1 + ||z|| + ||z'|)||z - z'||\end{aligned}$$

(L2) The map F^b satisfies the condition $(\mathbf{A}_\gamma)$.

(L3) The processes $F^d(\cdot,0,0)$ and $F^b(\cdot,0,0,0,)$ are uniformly bounded.

Remark 5.4.11. *One verifies that Assumption 5.4.10 implies Assumption 5.4.6 with $\lambda_t\Sigma_t(u) = \lambda_t J_t(u) = F^b(\cdot,0,0,u), (t,u) \in [0,T]\times\mathbb{R}$.*

The following proposition gives an estimate of the norm of a bounded solution to the RBSDE (5.50) w.r.t. its data. For a càdlàg process ζ, we set $\zeta^*_T = \sup_{t\in[0,T]}|\zeta_t|$.

Proposition 5.4.12. *Let (F,S,ξ) be a set of data satisfying Assumptions 5.4.3 and 5.4.10 and (Y,Z,U,K) be a bounded solution to the RBSDE(F,S,ξ). Then for every $q\geq 1$, there exist $r>1$ and a constant $C>0$ depending only $||Y||_\infty, |\lambda|_\infty, p$ and T such that*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^{2q} + \left(\int_0^T \left(||Z_s||^2 + |U_s|^2\lambda_s 1_{\{s\leq\tau\}}\right)ds\right)^q + K_T^{2q}\right] \\
&\leq\; C\left(\mathbb{E}\left[|\xi|^{2qr} + |S^*_T|^{2qr} + \left(\int_0^T |F(s,0,0,0)|ds\right)^{2qr}\right]\right)^{\frac{1}{r}}. \qquad (5.55)
\end{aligned}$$

For the proof, we will rely on two lemmas. First we introduce some notation before stating the lemmas. The equation describing the dynamics of Y can be written in the linearized form

$$-dY_s = (F(s,0,0,0) + \varphi_s Y_{s-} + \vartheta_s U_s)\,ds + dK_s - Z_s(dB^{\mathbb{G}}_s - \psi_s ds) - U_s dN^{\mathbb{G}}_s,\; s\in[0,T], \qquad (5.56)$$

where φ,ψ and ϑ are defined for $s\in[0,T]$ as follows

$$\varphi_s = \frac{F(s,Y_{s-},Z_s,U_s) - F(s,0,Z_s,U_s)}{Y_{s-}}1_{\{Y_{s-}\neq 0\}}, \qquad (5.57)$$

$$\vartheta_s = \frac{F(s,0,0,U_s) - F(s,0,0,0)}{U_s}1_{\{U_s\neq 0\}}, \qquad (5.58)$$

$$\psi_s = \frac{F(s,0,Z_s,U_s) - F(s,0,0,U_s)}{||Z_s||^2}Z_s 1_{\{||Z_s||\neq 0\}}. \qquad (5.59)$$

By (L1), we have $|\varphi_s|\leq C_y$ and $|\psi_s|\leq C_z(1+|Z_s|)$, $s\in[0,T]$. Thus $\psi\in\mathcal{H}^{2,n}_{BMO}(\mathbb{G},\mathbb{P})$ as $Z\in\mathcal{H}^{2,n}_{BMO}(\mathbb{G},\mathbb{P})$. We consider the probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ and with density

$$d\mathbb{Q}/d\mathbb{P}|_{\mathcal{G}_T} := \mathcal{E}\left(\psi\cdot B^{\mathbb{G}}\right)_T. \qquad (5.60)$$

We recall from Lemma 5.4.8 that U is pointwise bounded. Hence the condition $(\mathbf{A}_\gamma)$ implies that there exists a bounded process Π with norm depending on $||Y||_\infty$ such that

$$|\vartheta_s|\leq|\Pi_s|\lambda_s 1_{\{s\leq\tau\}}, s\in[0,T]. \qquad (5.61)$$

We begin with the following lemma which shows that $K\in\mathcal{IS}^p_T(\mathbb{G},\mathbb{Q})$ for every $p>1$.

Lemma 5.4.13. *We keep the notation and hypotheses of Proposition 5.4.12. Let $\mathbb{Q}$ be the measure defined by (5.60). Then for every $p>1$, there exists $C^{'}>0$ depending only on $||Y||_\infty, |\lambda|_\infty, p$ and T such that*

$$\mathbb{E}^{\mathbb{Q}}\left[K_T^{2p}\right] \;\leq\; C^{'}\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]}|Y_t|^{2p} + \left(\int_0^T \left(||Z_s||^2 + |U_s|^2\lambda_s 1_{\{s\leq\tau\}}\right)ds\right)^p + \left(\int_0^T |F(s,0,0,0)|ds\right)^{2p}\right].$$

Proof. Let $p > 1$. In the sequel, $C_1, C_2, \cdots$ denote arbitrary constants depending only on p, T, $||Y||_\infty$ and $|\lambda|_\infty$. Using (5.56), we have

$$K_T = Y_0 - Y_T - \int_0^T (\varphi_s Y_{s-} + \vartheta_s U_s + F(s,0,0,0))\, ds + \int_0^T U_s dN_s^{\mathbb{G}} + \int_0^T Z_s \left(dB_s^{\mathbb{G}} - \psi_s ds\right).$$

Observe that $|Y_0 - Y_T|^{2p} \le C_1 \sup_{t\in[0,T]} |Y_t|^{2p}$ and therefore

$$\begin{aligned} K_T^{2p} \le C_2 & \left(\sup_{t\in[0,T]} |Y_t|^{2p} + \left(\int_0^T |\varphi_s Y_{s-}| ds\right)^{2p} + \left(\int_0^T |\vartheta_s U_s| ds\right)^{2p} + \left(\int_0^T |F(s,0,0,0)| ds\right)^{2p}\right) \\ & + C_2 \left[\left(\int_0^T U_s dN_s^{\mathbb{G}}\right)^{2p} + \left(\int_0^T Z_s \left(dB_s^{\mathbb{G}} - \psi_s ds\right)\right)^{2p}\right]. \end{aligned}$$

Clearly $\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T |\varphi_s Y_{s-}| ds\right)^{2p}\right] \le C_3 \mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right]$. Recall from (5.61) that for $s \in [0,T]$ we have $|\vartheta_s| \le |\Pi_s| \lambda_s 1_{\{s\le\tau\}}$. An application of Hölder's inequality and the boundedness of Π and λ give

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T |\vartheta_s U_s| ds\right)^{2p}\right] & \le \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T |\Pi_s|^2 \lambda_s ds\right)^{p} \left(\int_0^T |U_s|^2 \lambda_s 1_{\{s\le\tau\}} ds\right)^{p}\right] \\ & \le C_4 \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T |U_s|^2 \lambda_s 1_{\{s\le\tau\}} ds\right)^{p}\right]. \end{aligned}$$

Since $N^{\mathbb{G}}$ and $B^{\mathbb{G}}$ are orthogonal, the process $N^{\mathbb{G}}$ is a $(\mathbb{G},\mathbb{Q})$-martingale. Moreover, $\left[N^{\mathbb{G}}, N^{\mathbb{G}}\right] = D$. Using the inequality $|U_s| \le 2\sup_{t\in[0,T]} |Y_t|$, $s \in [0,T]$ as well as the boundedness of Y, it follows from the Burkholder Davis-Gundy (BDG) inequalities that

$$\begin{aligned} \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T U_s dN_s^{\mathbb{G}}\right)^{2p}\right] & \le C_5 \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T |U_s|^2 d[N^{\mathbb{G}}, N^{\mathbb{G}}]_s\right)^{p}\right] = C_5 \mathbb{E}^{\mathbb{Q}}\left[\int_0^T |U_s|^{2p} dD_s\right] \\ & = C_5 \mathbb{E}^{\mathbb{Q}}\left[\int_0^T |U_s|^{2p} dN_s^{\mathbb{G}} + \int_0^T |U_s|^{2p} \lambda_s 1_{\{s\le\tau\}} ds\right] \le C_6 \mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]} |Y_t|^{2p}\right]. \end{aligned}$$

By Girsanov's theorem, $B^{\mathbb{G}} - \int_0^{\cdot} \psi_s ds$ is a $(\mathbb{G},\mathbb{Q})$-martingale. It follows from the BDG inequalities that

$$\mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T Z_s (dB_s^{\mathbb{G}} - \psi_s ds)\right)^{2p}\right] \le C_7 \mathbb{E}^{\mathbb{Q}}\left[\left(\int_0^T ||Z_s||^2 ds\right)^{p}\right].$$

Combining all these estimations, we obtain the desired inequality. □

The following lemma provides a conditional estimate of the norm of the triplet (Y, Z, U).

Lemma 5.4.14. *We keep the notation and hypotheses of Proposition 5.4.12. Let $\mathbb{Q}$ be the measure defined by (5.60). Assume that there exists a constant $\beta > 0$ such that for every $t \in [0,T]$*

$$\begin{aligned} & e^{\beta t} Y_t^2 + \int_t^T \left(||Z_s||^2 + \frac{1}{2}|U_s|^2 \lambda_s 1_{\{s\le\tau\}}\right) ds \\ \le\ & e^{\beta T}|\xi|^2 + 2e^{\beta T} K_T |S_T^*| + 2\int_t^T e^{\beta s}|Y_{s-}||F(s,0,0,0)| ds \qquad (5.62) \\ & -2\int_t^T e^{\beta s} Y_{s-} Z_s (dB_s^{\mathbb{G}} - \psi_s ds) - \int_t^T e^{\beta s} U_s (2Y_{s-} + U_s) dN_s^{\mathbb{G}}. \end{aligned}$$

Then for every $p>1$, there exists a constant C_p depending only on $||Y||_\infty, |\lambda|_\infty, p$ and T such that

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]}|Y_t|^{2p}+\left(\int_0^T\left(||Z_s||^2+|U_s|^2\lambda_s 1_{\{s\leq\tau\}}\right)ds\right)^p\right]$$
$$\leq C_p\mathbb{E}^{\mathbb{Q}}\left[|\xi|^{2p}+K_T^p|S_T^*|^p+\left(\int_0^T|F(s,0,0,0)|ds\right)^{2p}\right]. \tag{5.63}$$

Proof. The inequality (5.62) is identical to the inequality (12) in [ABSEL10] and the proof of (5.63) follows from Lemma 2.4 in [ABSEL10]. □

Proof of Proposition 5.4.12. Let $q\geq 1$. The proof will consist of two steps.
Step 1. We first establish (5.55) under the equivalent measure $\mathbb{Q}$, i.e. we show that for every $p>1$ there exists a constant C_p' depending only $||Y||_\infty, |\lambda|_\infty, p$ and T such that

$$\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]}|Y_t|^{2p}+\left(\int_0^T\left(||Z_s||^2+|U_s|^2\lambda_s 1_{\{s\leq\tau\}}\right)ds\right)^p+K_T^{2p}\right]$$
$$\leq C_p'\mathbb{E}^{\mathbb{Q}}\left[|\xi|^{2p}+|S_T^*|^{2p}+\left(\int_0^T|F(s,0,0,0)|ds\right)^{2p}\right]. \tag{5.64}$$

Let $\beta>0$ be a constant to be chosen later. An application of Itô's formula yields for $t\in[0,T]$

$$e^{\beta t}Y_t^2+\int_t^T e^{\beta s}||Z_s||^2ds+\sum_{t<s\leq T}e^{\beta s}|\delta Y_s|^2=e^{\beta T}|\xi|^2-\beta\int_t^T e^{\beta s}Y_{s-}^2ds-2\int_t^T Y_{s-}dY_s. \tag{5.65}$$

Clearly, $\Delta Y_s=U_s\Delta N_s^{\mathbb{G}}+\Delta K_s, s\in[0,T]$. Moreover the jump times of K and $N^{\mathbb{G}}$ are distinct. Hence for $t\in[0,T]$

$$\sum_{t<s\leq T}e^{\beta s}|\Delta Y_s|^2 = \sum_{t<s\leq T}e^{\beta s}|\Delta K_s|^2+\sum_{t<s\leq T}e^{\beta s}|U_s|^2|\Delta N_s^{\mathbb{G}}|^2\geq\sum_{t<s\leq T}e^{\beta s}|U_s|^2|\Delta N_s^{\mathbb{G}}|^2.$$

Recall that $dD_t=dN_t^{\mathbb{G}}+\lambda_t 1_{\{t\leq\tau\}}dt,\ t\in[0,T]$ and $N^{\mathbb{G}}$ has jump size 1. Thus $|\Delta N_s^{\mathbb{G}}|^2=|\Delta N_s^{\mathbb{G}}|, s\in[0,T]$. As $|U|\leq 2||Y||_\infty$, the process $\int_0^\cdot e^{\beta s}|U_s|^2dD_s$ is a special semimartingale and for $t\in[0,T]$

$$\sum_{t<s\leq T}e^{\beta s}|U_s|^2|\Delta N_s^{\mathbb{G}}|^2 = \int_t^T e^{\beta s}|U_s|^2dD_s=\int_t^T e^{\beta s}|U_s|^2dN_s^{\mathbb{G}}+\int_t^T e^{\beta s}|U_s|^2\lambda_s 1_{\{s\leq\tau\}}ds.$$

We infer from (5.65) that for $t\in[0,T]$

$$e^{\beta t}Y_t^2+\int_t^T e^{\beta s}\left(||Z_s||^2+|U_s|^2\lambda_s 1_{\{s\leq\tau\}}\right)ds$$
$$\leq e^{\beta T}|\xi|^2-\beta\int_t^T e^{\beta s}|Y_{s-}|^2ds-2\int_t^T e^{\beta s}Y_{s-}dY_s-\int_t^T e^{\beta s}|U_s|^2dN_s^{\mathbb{G}}. \tag{5.66}$$

From the Skorohod condition $\int_0^T(Y_{s-}-S_{s-})dK_s=0$, we deduce that

$$\int_t^T e^{\beta s}Y_{s-}dK_s = \int_t^T e^{\beta s}(Y_{s-}-S_{s-})dK_s+\int_t^T e^{\beta s}S_{s-}dK_s\leq\int_t^T e^{\beta s}S_{s-}dK_s,$$
$$\leq 2e^{\beta T}|S_T^*|K_T,\ t\in[0,T].$$

By the above bound, it follows from (5.66) and the equation (5.56) that for $t \in [0,T]$

$$
\begin{aligned}
e^{\beta t} Y_t^2 + \int_t^T e^{\beta s}\left(||Z_s||^2 + |U_s|^2 \lambda_s 1_{\{s\le\tau\}}\right) ds \\
\le e^{\beta T}|\xi|^2 + 2e^{\beta T}|S_T^*|K_T + 2\int_t^T e^{\beta s}|Y_{s-}||F(s,0,0,0)|ds \\
+ \int_t^T e^{\beta s}\left(-\beta|Y_{s-}|^2 + 2\varphi_s Y_{s-}^2 + 2\vartheta_s Y_{s-}U_s\right) \\
-2\int_t^T e^{\beta s} Y_{s-} Z_s \left(dB_s^{\mathbb{G}} - \psi_s ds\right) - 2\int_t^T e^{\beta s} U_s\left(2Y_{s-} + U_s\right) dN_s^{\mathbb{G}}.
\end{aligned}
\tag{5.67}
$$

Note that for $s \in [0,T], |\vartheta_s| \le \Pi_s|\lambda_s|1_{\{s\le\tau\}}$ and Young's inequality $2|ab| \le 2a^2 + \frac{1}{2}b^2, a,b \in \mathbb{R}$ yields $2|\vartheta_s Y_{s-} U_s| \le 2\Pi_s^2|Y_{s-}|^2\lambda_s + \frac{1}{2}|U_s|^2\lambda_s 1_{\{s\le\tau\}}$. Choosing $\beta \ge 2||\varphi||_\infty + 2||\Pi||_\infty|\lambda|_\infty$ and rearranging terms in (5.67), one sees that (5.62) is satisfied, i.e. for $t \in [0,T]$

$$
\begin{aligned}
e^{\beta t} Y_t^2 + \int_t^T \left(||Z_s||^2 + \frac{1}{2}|U_s|^2\lambda_s 1_{\{s\le\tau\}}\right) ds \\
\le \quad e^{\beta T}|\xi|^2 + 2e^{\beta T}|S_T^*|K_T + 2\int_t^T e^{\beta s}|Y_{s-}||F(s,0,0,0)|ds \\
-2\int_t^T e^{\beta s} Y_{s-} Z_s (dB_s^{\mathbb{G}} - \psi_s ds) - \int_t^T e^{\beta s} U_s (2Y_{s-} + U_s) dN_s^{\mathbb{G}}.
\end{aligned}
$$

Let $p > 1$. By Lemma 5.4.14, there exists $C_p > 0$ depending only on $||Y||_\infty, |\lambda|_\infty, p$ and T such that

$$
\begin{aligned}
\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]}|Y_t|^{2p} + \left(\int_0^T \left(||Z_s||^2 + |U_s|^2\lambda_s 1_{\{s\le\tau\}}\right) ds\right)^p\right] \\
\le C_p \mathbb{E}^{\mathbb{Q}}\left[|\xi|^{2p} + |S_T^*|^p K_T^p + \left(\int_0^T |F(s,0,0,0)|ds\right)^{2p}\right].
\end{aligned}
\tag{5.68}
$$

Now Lemma 5.4.13 gives an estimate of $\mathbb{E}^{\mathbb{Q}}\left[K_T^{2p}\right]$ which combined with (5.68) yields

$$
\mathbb{E}^{\mathbb{Q}}\left[K_T^{2p}\right] \quad \le \quad \overline{C}\mathbb{E}^{\mathbb{Q}}\left[|\xi|^{2p} + |S_T^*|^p K_T^p + \left(\int_0^T |F(s,0,0,0)|ds\right)^{2p}\right], \tag{5.69}
$$

for some constant $\overline{C} > 0$. Observe from (5.68) and (5.69) that (5.64) holds, provided we have an appropriate bound for $\mathbb{E}^{\mathbb{Q}}\left[K_T^p|S_T^*|^p\right]$. Applying Hölder's inequality and binomial inequality, we have

$$
\overline{C}\mathbb{E}^{\mathbb{Q}}\left[|S_T^*|^p K_T^p\right] \quad \le \quad \frac{1}{2}\overline{C}^2\mathbb{E}^{\mathbb{Q}}\left[|S_T^*|^{2p}\right] + \frac{1}{2}\mathbb{E}^{\mathbb{Q}}\left[K_T^{2p}\right]. \tag{5.70}
$$

Combining (5.69) and (5.70), we see that there exists a constant $C' > 0$ such that

$$
\mathbb{E}^{\mathbb{Q}}\left[K_T^p|S_T^*|^p\right] \le C'\mathbb{E}^{\mathbb{Q}}\left[|\xi|^{2p} + |S_T^*|^{2p} + \left(\int_0^T |F(s,0,0,0)|ds\right)^{2p}\right].
$$

Inserting the latter inequality into (5.68) and (5.69), one obtains (5.64).
Step 2. We show (5.55) using (5.64) and Bayes' formula. Recall from (5.60) that $d\mathbb{Q}/d\mathbb{P}|_{\mathcal{G}_T} = \mathcal{E}\left(\psi\cdot B^{\mathbb{G}}\right)_T$. Define $\widehat{B}^{\mathbb{G}} = B^{\mathbb{G}} - \int_0^\cdot \psi_s ds$. One verifies that $\mathcal{E}\left(\psi\cdot B^{\mathbb{G}}\right)_T \mathcal{E}\left(-\psi\cdot\widehat{B}^{\mathbb{G}}\right)_T = 1$.

The process $\psi\cdot B^{\mathbb{G}}$ being a BMO martingale, we infer from [Kaz94, Theorems 3.1 and 3.6] that there exists $p,l>1$ depending only on $||\psi||_{\mathcal{H}^{2,n}_{BMO}(\mathbb{G},\mathbb{P})}$ such that $\mathcal{E}\left(\psi\cdot B^{\mathbb{G}}\right)_T\in L^p(\mathcal{G}_T,\mathbb{P})$ and $\mathcal{E}\left(-\psi\cdot \widehat{B}^{\mathbb{G}}\right)_T\in L^l(\mathcal{G}_T,\mathbb{Q})$. Let p' and $l'>1$ be such that $\frac{1}{p}+\frac{1}{p'}=1$ and $\frac{1}{l}+\frac{1}{l'}=1$. In what follows, $C>0$ is an arbitrary constant that changes values from one line to the other. Applying Bayes' formula, Hölder's inequality and binomial inequality together with (5.64), we obtain with $r=p'l'$ that

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]}|Y_t|^{2q}\right] &= \mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}\left(-\psi\cdot\widehat{B}^{\mathbb{G}}\right)_T\sup_{t\in[0,T]}|Y_t|^{2q}\right]\\
&\le\left(\mathbb{E}^{\mathbb{Q}}\left[\mathcal{E}\left(-\psi\cdot\widehat{B}^{\mathbb{G}}\right)^l_T\right]\right)^{\frac{1}{l}}\left(\mathbb{E}^{\mathbb{Q}}\left[\sup_{t\in[0,T]}|Y_t|^{2ql'}\right]\right)^{\frac{1}{l'}}\\
&\le C\left(\mathbb{E}^{\mathbb{Q}}\left[|\xi|^{2ql'}+|S^*_T|^{2ql'}+\left(\int_0^T|F(s,0,0,0)|ds\right)^{2ql'}\right]\right)^{\frac{1}{l'}}\\
&\le C\left(\mathbb{E}\left[\mathcal{E}\left(\psi\cdot B^{\mathbb{G}}\right)^p_T\right]\right)^{\frac{1}{rpl'}}\left(\mathbb{E}\left[|\xi|^{2qr}+|S^*_T|^{2qr}+\left(\int_0^T|F(s,0,0,0)|ds\right)^{2qr}\right]\right)^{\frac{1}{r}}.
\end{aligned}
$$

One shows analogously that

$$
\begin{aligned}
&\mathbb{E}\left[\left(\int_0^T\left(||Z_s||^2+|U_s|^2\lambda_s 1_{\{s\le\tau\}}\right)ds\right)^q+K_T^{2q}\right]\\
&\le C\left(\mathbb{E}\left[|\xi|^{2qr}+|S^*_T|^{2qr}+\left(\int_0^T|F(s,0,0,0)|ds\right)^{2qr}\right]\right)^{\frac{1}{r}}.
\end{aligned}
$$

This completes the proof. □

The following theorem provides a priori estimates of solutions to the RBSDE (5.50) w.r.t. its input data.

Theorem 5.4.15. *Let (F,S,ξ) and $(\overline{F},\overline{S},\overline{\xi})$ be two sets of data satisfying Assumptions 5.4.3 and 5.4.10. Let (Y,Z,U,K) (resp. $(\overline{Y},\overline{Z},\overline{U},\overline{K})$ be bounded solutions to RBSDE(F,S,ξ) (resp. $(\overline{F},\overline{S},\overline{\xi})$). Let $\delta Y,\delta Z,\delta U,\delta K,\delta F,\delta\xi$ and δS be defined by:*

$$
\begin{aligned}
&\delta Y=Y-\overline{Y},\quad \delta Z=Z-\overline{Z},\quad \delta U=U-\overline{U},\quad \delta K=K-\overline{K},\\
&\delta F=F(,\overline{Y}_-,\overline{Z},\overline{U})-\overline{F}(\cdot,\overline{Y}_-,\overline{Z},\overline{U}),\quad \delta\xi=\xi-\overline{\xi},\quad \delta S=S-\overline{S}.
\end{aligned}
$$

For every $q\ge1$, there exists a constant $C'>0$ and $r>1$ depending only $T,|\lambda|_\infty,||Y||_\infty$ and $||\overline{Y}||_\infty$ such that

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{t\in[0,T]}|\delta Y_t|^{2q}+\left(\int_0^T\left(|\delta Z_s|^2+|\delta U_s|^2\lambda_s 1_{\{\tau\le s\}}\right)ds\right)^q+|\delta K_T|^{2q}\right]\\
&\le C'\left(\mathbb{E}\left[|\delta\xi|^{2qr}+\sup_{t\in[0,T]}|\delta S_t|^{qr}|K_T+\overline{K}_T|^{qr}+\left(\int_0^T|\delta F_s|ds\right)^{2qr}\right]\right)^{\frac{1}{r}}.
\end{aligned}
\tag{5.71}
$$

Proof. For the proof, we will proceed as in Proposition 5.4.12 by first providing a similar estimate under an auxiliary measure (see (5.74)) and obtain (5.71) using Bayes' formula and Hölder's inequalities. In order to introduce the aforementioned measure, let us fix some notation. Let $\overline{\varphi}, \overline{\psi}$ and $\overline{\vartheta}$ be the processes defined for $s \in [0,T]$ by

$$
\begin{aligned}
\overline{\varphi}_s &= \frac{F(s,Y_{s-},Z_s,U_s) - F(s,\overline{Y}_{s-},Z_s,U_s)}{\delta Y_{s-}} 1_{\{\delta Y_s \neq 0\}},\\
\overline{\vartheta}_s &= \frac{F(s,\overline{Y}_{s-},\overline{Z}_s,U_s) - F(s,\overline{Y}_s,\overline{Z}_s,U_s)}{\delta U_s} 1_{\{\delta U_s \neq 0\}},\\
\overline{\psi}_s &= \frac{F(s,\overline{Y}_{s-},Z_s,U_s) - F(s,\overline{Y}_{s-},\overline{Z}_s,U_s)}{|\delta Z_s|^2} (\delta Z_s)\, 1_{\{||\delta Z_s|| \neq 0\}}.
\end{aligned}
$$

One verifies that for $t \in [0,T]$

$$
-F(t,Y_{t-},Z_t,U_t) + \overline{F}(t,\overline{Y}_{t-},\overline{Z},\overline{U}_t) \;=\; -\overline{\varphi}_t \delta Y_t - \overline{\psi}_t \delta Z_t - \vartheta_t \delta U_t - \delta F_t.
$$

Using the definition of solutions, we have for $t \in [0,T]$

$$
d\delta Y_t \;=\; -\left(\overline{\varphi}_t \delta Y_{t-} + \overline{\vartheta}_t \delta U_t\right) dt - \delta F_t dt - d\delta K_t + \delta Z_t \left(dB_t^{\mathbb{G}} - \overline{\psi}_t dt\right) + \delta U_t dN_t^{\mathbb{G}}. \tag{5.72}
$$

By Lemma 5.4.8, δU is bounded. Due to the condition $(\mathbf{A}_\gamma)$, there exists a real valued $\mathbb{F}$-predictable bounded process $\overline{\Pi}$ with norm depending on $||Y||_\infty, ||\overline{Y}||_\infty$ such that

$$
|\overline{\vartheta}_t| \leq |\overline{\Pi}_t| \lambda_t 1_{\{t \leq \tau\}},\ t \in [0,T].
$$

By Assumption 5.4.10, $|\overline{\varphi}| \leq C_y$ and $|\overline{\psi}| \leq C_z(1 + |\delta Z|)$. Clearly $\overline{\psi} \in \mathcal{H}^{2,n}_{BMO}(\mathbb{G},\mathbb{P})$ since $\delta Z \in \mathcal{H}^{2,n}_{BMO}(\mathbb{G},\mathbb{P})$. We consider the measure $\overline{\mathbb{Q}}$ equivalent to $\mathbb{P}$ on $\mathcal{G}_T$ with density

$$
d\overline{\mathbb{Q}}/d\mathbb{P}|_{\mathcal{G}_T} = \mathcal{E}\left(\overline{\psi} \cdot B^{\mathbb{G}}\right)_T. \tag{5.73}
$$

To show (5.71), it will suffice to show that for every $p > 1$, there exists a constant C_p which depends only on $p, T, |\lambda|_\infty, ||Y||_\infty$ and $||\overline{Y}||_\infty$ such that

$$
\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{Q}}}\left[\sup_{t\in[0,T]} |\delta Y_t|^{2p} + \left(\int_0^T \left(|\delta Z_s|^2 + |\delta U_s|^2 \lambda_s 1_{\{s \leq \tau\}}\right) ds\right)^p + \delta K_T^{2p}\right]\\
&\leq C_p \mathbb{E}^{\overline{\mathbb{Q}}}\left[|\delta \xi|^{2p} + \sup_{t\in[0,T]} |\delta S_t|^p |K_T + \overline{K}_T|^p + \left(\int_0^T |\delta F_s| ds\right)^{2p}\right].
\end{aligned} \tag{5.74}
$$

Indeed, (5.74) and the same arguments as those in **Step 2** of Proposition 5.4.12 lead to (5.71). Let $p > 1$. In the sequel, we focus on showing (5.74). Let $\beta > 0$ to be chosen later. Let $t \in [0,T]$. Then by Itô's formula, we have

$$
e^{\beta T}|\delta Y_t|^2 + \int_t^T e^{\beta s} |\delta Z_s|^2 ds + \sum_{t<s\leq T} e^{\beta s} |\Delta \delta Y_s|^2 = e^{\beta T}|\delta \xi|^2 - \beta \int_t^T e^{\beta s} \delta Y_{s-}^2 ds - 2\int_t^T e^{\beta s} \delta Y_{s-} d\delta Y_s.
$$

Applying the same arguments as those leading to (5.66), we obtain

$$
e^{\beta t}|\delta Y_t|^2 + \int_t^T e^{\beta s}\left(||Z_s||^2 + |\delta U_s|^2 \lambda_s 1_{\{s\leq\tau\}}\right) ds \tag{5.75}
$$

$$
\leq e^{\beta T}|\delta Y_T|^2 - \beta \int_t^T e^{\beta s} \delta Y_{s-}^2 ds - 2\int_t^T e^{\beta s} \delta Y_{s-} d\delta Y_s - \int_t^T e^{\beta s} |\delta U_s|^2 dN_s^{\mathbb{G}}. \tag{5.76}
$$

Note that the Skorohod conditions $\int_0^T (Y_{s-} - S_{s-}) dK_s = 0$ and $\int_0^T (\overline{Y}_{s-} - \overline{S}_-) d\overline{K}_s = 0$, and the monotonicity of K and $\overline{K}$ entail that $\int_t^T e^{\beta s}(\delta Y_{s-} - \delta S_-) d\delta K_s \leq 0$. Consequently

$$
\int_t^T e^{\beta s} \delta Y_{s-} d\delta K_s \;\leq\; \int_t^T e^{\beta s} \delta S_{s-} d\delta K_s \leq e^{\beta T} \sup_{s\in[0,T]} |\delta S_s| \times |K_T + \overline{K}_T|. \tag{5.77}
$$

Inserting (5.77) into (5.75) and using the dynamical description (5.72) of δY leads to the inequality

$$
\begin{aligned}
&e^{\beta t}|\delta Y_t|^2 + \int_t^T e^{\beta s}\left(|\delta Z|_s^2 + |\delta U_s|^2\lambda_s 1_{\{s\le\tau\}}\right) ds \\
&\le e^{\beta T}|\delta\xi|^2 + 2e^{\beta T}\sup_{t\in[0,T]}|\delta S_t| \times |K_T + \overline{K}_T| + 2\int_t^T e^{\beta s}|Y_{s-}|\delta F_s| ds \\
&+ \int_t^T e^{\beta s}\left(-\beta|Y_{s-}|^2 + 2\overline{\varphi}_{s-}\delta Y_{s-}^2 + 2\overline{\vartheta}_s\delta Y_{s-}\delta U_s\right) ds \\
&-2\int_t^T e^{\beta s}\delta Y_{s-}\delta Z_s\left(dB_s^{\mathbb{G}} - \overline{\psi}_s ds\right) - 2\int_t^T e^{\beta s}\delta U_s\left(2\delta Y_{s-} + \delta U_s\right) dN_s^{\mathbb{G}}.
\end{aligned}
$$

Employing the binomial inequality $2|\overline{\vartheta}\delta Y_{s-}\delta U_s| \le 2|\overline{\Pi}_s|^2|\delta Y_{s-}^2\lambda_s + \frac{1}{2}|\delta U_s|^2\lambda_s 1_{\{s\le\tau\}},\ s\in[0,T]$ and choosing $\beta \ge 2||\overline{\varphi}||_\infty + 2||\overline{\Pi}||_\infty|\lambda|_\infty$, we obtain

$$
\begin{aligned}
&e^{\beta t}|\delta Y_t|^2 + \int_t^T e^{\beta s}\left(|\delta Z|_s^2 + |\delta U_s|^2\lambda_s 1_{\{s\le\tau\}}\right) ds \\
&\le e^{\beta T}|\delta\xi|^2 + 2e^{\beta T}\sup_{t\in[0,T]}|\delta S_t| \times |K_T + \overline{K}_T| + 2\int_t^T e^{\beta s}|Y_{s-}| \times |\delta F_s| ds \\
&-2\int_t^T e^{\beta s}\delta Y_{s-}\delta Z_s\left(dB_s^{\mathbb{G}} - \overline{\psi}_s ds\right) - 2\int_t^T e^{\beta s}\delta U_s\left(2\delta Y_{s-} + \delta U_s\right) dN_s^{\mathbb{G}}.
\end{aligned}
$$

The above inequality is of type (5.62) and Lemma 5.4.14 guarantees that there exists $C_p^{'} > 0$ such that

$$
\begin{aligned}
&\mathbb{E}^{\overline{\mathbb{Q}}}\left[\sup_{t\in[0,T]}|\delta Y_t|^{2p} + \left(\int_0^T\left(|\delta Z_s|^2 + |\delta U_s|^2\lambda_s 1_{\{s\le\tau\}}\right) ds\right)^p\right] \\
&\le C_p^{'}\mathbb{E}^{\overline{\mathbb{Q}}}\left[|\delta\xi|^{2p} + \sup_{t\in[0,T]}|\delta S_t|^p \times |K_T + \overline{K}_T|^p + \left(\int_0^T |\delta F_s| ds\right)^{2p}\right].
\end{aligned}
$$

It remains to give a suitable estimate of $\mathbb{E}^{\overline{\mathbb{Q}}}\left[\delta K_T^{2p}\right]$. Using (5.72), we have

$$
\delta K_T \;=\; \delta Y_0 - \delta\xi - \int_0^T\left(\overline{\varphi}_s\delta Y_{s-} + \overline{\vartheta}_s\delta U_s + \delta F_s\right) ds + \int_0^T \delta Z_s\left(dB_s^{\mathbb{G}} - \overline{\psi}_s ds\right) + \int_0^T \delta U_s dN_s^{\mathbb{G}}.
$$

Employing similar arguments as those in the proof of Lemma 5.4.13, one shows that

$$
\mathbb{E}^{\overline{\mathbb{Q}}}\left[\delta K_T^{2p}\right] \le C_5\mathbb{E}^{\overline{\mathbb{Q}}}\left[\sup_{t\in[0,T]}|\delta Y_t|^{2p} + \left(\int_0^T\left(|\delta Z_s|^2 + |\delta U_s|^2\lambda_s 1_{\{s\le\tau\}}\right) ds\right)^p + \left(\int_0^T |\delta F_s| ds\right)^{2p}\right].
$$

We obtain (5.74) by summing up the last two estimates. Following the same arguments as in the proof of **Step 2** of Proposition 5.4.12, (5.71) is seen to be a consequence of (5.74). □

Theorem 5.4.15 leads to the following uniqueness result.

Corollary 5.4.16. *Suppose that Assumptions 5.4.3 and 5.4.10 hold. Then the RBSDE (5.50) admits at most one bounded solution* (Y, Z, U, K).

5.4.2 Existence of solutions

In the previous section, we derived a stability result under Assumptions 5.4.3 and 5.4.10 which guarantees the uniqueness of bounded solutions to the RBSDE(F, S, ξ). We focus in this section on proving the existence of a bounded solution and this under the weaker hypotheses Assumptions 5.4.3 and 5.4.6. To achieve this, we will apply the two step decomposition approach given by Theorem 5.3.5. To this end, it suffices to show that the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}\left(F^{\mathrm{d}}, S^{\mathrm{d}}(\tau), \xi^{\mathrm{d}}(\tau)\right)$ and RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}\left(F^{\mathrm{b}}_\Gamma, S^{\mathrm{b}}, \xi^{\mathrm{b}}\right)$ admit a bounded solution for Γ appropriately chosen. First let us make precise the structure of the solution of the aforementioned RBSDEs as well as their reformulation in the context of Assumption 5.4.1. Following Part ii) of Proposition 5.4.2, the filtration $\mathbb{G}^\tau$ is continuous and every local martingale in the space $\mathcal{M}^{loc}_{\mathbb{G}^\tau}(B^{\mathbb{G}^\tau}, \mathbb{P})$ is the null martingale. Consequently, every solution to RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}\left(F^{\mathrm{d}}, S^{\mathrm{d}}(\tau), \xi^{\mathrm{d}}(\tau)\right)$ is of the form $\left(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), 0, K^{\mathrm{d}}(\tau)\right)$ with $\left(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau)\right)$ satisfying

$$\begin{cases} Y^{\mathrm{d}}_t(\tau) = \xi^{\mathrm{d}}(\tau) + \int_t^T F^{\mathrm{d}}(s, Y^{\mathrm{d}}_{s-}(\tau), Z^{\mathrm{d}}_s(\tau))ds + \int_t^T dK^{\mathrm{d}}_s(\tau) - \int_t^T Z^{\mathrm{d}}_s(\tau)dB^{\mathbb{G}^\tau}_s, \ t \in [0, T], \\ Y^{\mathrm{d}}_t(\tau) \geq S^{\mathrm{d}}_t(\tau), \ t \in [0, T], \\ \int_0^T (Y^{\mathrm{d}}_t(\tau) - S^{\mathrm{d}}_t(\tau))dK^{\mathrm{d}}_t(\tau) = 0. \end{cases} \tag{5.78}$$

We will also refer to (5.78) as the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}\left(F^{\mathrm{d}}, S^{\mathrm{d}}(\tau), \xi^{\mathrm{d}}(\tau)\right)$ and a solution will simply be denoted by the triplet $\left(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau)\right)$.

Remark 5.4.17. *As mentioned in Remark 5.3.10, working with the data $(F^d, S^d(\tau), \xi^d(\tau))$ leads to the same result as with the data $(F^dD_-, S^d(\tau)D, \xi^d(\tau)D_T)$ when carrying out the pasting procedure since we are only interested in the values of the solution after τ. We choose the data $(F^d, S^d, \xi^d(\tau))$ as we aim to apply Theorem 5.4.18 below to ensure the existence of a solution to RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}\left(F^d, S^d(\tau), \xi^d(\tau)\right)$ under Assumption 5.4.6. The Skorohod condition in (5.78) is equivalent to that in (5.17). This follows from the continuity of the obstacle $S^d(\tau)$ and the first component of every solution to (5.78), see Theorem 5.4.18.*

We now move to the pre-default RBSDE. Let Γ be an $\mathbb{F}$-optional process. Recall from (5.18) that

$$F^{\mathrm{b}}_\Gamma(t, \cdot, y, z) \quad = \quad F^{\mathrm{b}}(t, \cdot, y, z, \Gamma_t - y) + \lambda_t(\Gamma_t - y), \ (t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^n. \tag{5.79}$$

Since B admits the predictable representation property w.r.t. $(\mathbb{F}, \mathbb{P})$-local martingales, Theorem 13.12 in [HWY92] implies that every $(\mathbb{F}, \widehat{\mathbb{Q}})$-local martingale is a stochastic integral w.r.t. the process $B^{\mathbb{F}}$ given by (5.15). As a result, the null martingale is the only element of the space $\mathcal{M}^{loc}_{\mathbb{F}}(B^{\mathbb{F}}, \widehat{\mathbb{Q}})$. Hence every solution to the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_\Gamma, S^{\mathrm{b}}, \xi^{\mathrm{b}})$ is of the form $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, 0, K^{\mathrm{b}})$ where $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, K^{\mathrm{b}})$ satisfies

$$\begin{cases} Y^{\mathrm{b}}_t = \xi^{\mathrm{b}} + \int_t^T F^{\mathrm{b}}_\Gamma(s, Y^{\mathrm{b}}_s, Z^{\mathrm{b}}_s)ds + \int_t^T dK^{\mathrm{b}}_s - \int_t^T Z^{\mathrm{b}}_s dB^{\mathbb{F}}_s, \ t \in [0, T], \\ Y^{\mathrm{b}}_t \geq S^{\mathrm{b}}_t, \ t \in [0, T], \\ \int_0^T (Y^{\mathrm{b}}_t - S^{\mathrm{b}}_t)dK^{\mathrm{b}}_t = 0. \end{cases} \tag{5.80}$$

We will refer to (5.80) as RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_\Gamma, S^{\mathrm{b}}, \xi^{\mathrm{b}})$ and we will denote a solution by $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, K^{\mathrm{b}})$.

To show the existence of a solution to both RBSDEs (5.78) and (5.80), we will use the following result:

Theorem 5.4.18. *[KLQT02, Theorem 2.1] Let $X \in \mathcal{S}^\infty_T(\mathbb{F})$ with continuous paths and $\zeta \in L^\infty(\mathcal{F}_T)$. Let $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, be $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^n) - \mathcal{B}(\mathbb{R})$-measurable. Suppose that the following hold:*

1. *X has continuous paths and $X_T \leq \zeta$,*
2. *for all $(t,\omega) \in [0,T] \times \Omega$, $f(t,\omega,\cdot,\cdot)$ is continuous,*
3. *there exists a positive constant $C > 0$ and a strictly positive function l such that*

$$\int_0^\infty \frac{dx}{l(x)} = +\infty \text{ with } |f(t,\omega,y,z)| \leq l(y) + C||z||^2,\ dt \otimes \mathbb{P}\text{-a.e. } (t,\omega) \in [0,T] \times \Omega.$$

Then the RBSDE in the filtration $\mathbb{F}$ and w.r.t. the measure $\mathbb{P}$ associated to the driver f, obstacle X and terminal value ζ (hereby denoted by RBSDE$_{(\mathbb{F},\mathbb{P})}(f,X,\zeta)$) admits a solution $(P,Q,V) \in \mathcal{S}_T^\infty(\mathbb{F}) \times \mathcal{H}_T^{2,n}(\mathbb{F},\mathbb{P}) \times \mathcal{IS}_T(\mathbb{F})$, i.e. (P,Q,V) satisfies

$$\begin{cases} P_t &= \zeta + \int_t^T f(s,P_s,Q_s)ds + V_T - V_t - \int_t^T Q_s dB_s,\ t \in [0,T], \\ P_t &\geq X_t \text{ and } \int_0^T (P_s - X_s)\, dV_s = 0. \end{cases} \tag{5.81}$$

The following proposition from [Lio14] collects additional properties of solutions for RBSDEs in continuous filtrations.

Proposition 5.4.19. *Let (f,X,ζ) be as in Theorem 5.4.18 and $(P,Q,V) \in \mathcal{S}_T^\infty(\mathbb{F}) \times \mathcal{H}_T^{2,n}(\mathbb{F},\mathbb{P}) \times \mathcal{IS}_T(\mathbb{F})$ a solution to the RBSDE$_{(\mathbb{F},\mathbb{P})}(f,X,\zeta)$. Then $(Q,V) \in \mathcal{H}_{BMO}^{2,n}(\mathbb{F},\mathbb{P}) \times \mathcal{IS}_{BMO}(\mathbb{F},\mathbb{P})$.*

Assume additionally that f satisfies Assumption 5.4.10. Let $(\overline{f},\overline{X},\overline{V})$ be another set of data satisfying the conditions in Theorem 5.4.18 and $(\overline{P},\overline{Q},\overline{V}) \in \mathcal{S}_T^\infty(\mathbb{F}) \times \mathcal{H}_T^{2,n}(\mathbb{F},\mathbb{P}) \times \mathcal{IS}_{BMO}(\mathbb{F},\mathbb{P})$ a solution to the RBSDE$_{(\mathbb{F},\mathbb{P})}(\overline{f},\overline{X},\overline{\zeta})$. If $f(\cdot,\overline{P},\overline{Q}) \leq \overline{f}(\cdot,\overline{P},\overline{Q})$, $dt \otimes \mathbb{P}$-a.e.,$X \leq \overline{X}$ and $\zeta \leq \overline{\zeta}$, then $P \leq \overline{P}$.

Proof. Apply Proposition 1 in [Lio14] for the first assertion and Theorem 3 in [Lio14] for the second one. ☐

Remark 5.4.20. *Theorem 5.4.18 and Proposition 5.4.19 are stated in the reference filtration $\mathbb{F}$ and the reference measure $\mathbb{P}$. They hold for every continuous filtration for which a martingale representation theorem is valid.*

Note that due to Assumption 5.4.3 and Assumption 5.4.6 (Q1), the triplet $(F^{\mathrm{d}}, S^{\mathrm{d}}(\tau), \xi^{\mathrm{d}}(\tau))$ satisfies all the conditions of Theorem 5.4.18. We therefore have the following result.

Theorem 5.4.21. *Suppose that Assumptions 5.4.3 and 5.4.6 hold. Then there exists a bounded solution $(Y^d(\tau), Z^d(\tau), K^d(\tau))$ to RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^d, S^d(\tau), \xi^d(\tau))$. Moreover, $(Z^d(\tau), K^d(\tau)) \in \mathcal{H}_{BMO}^{2,n}(\mathbb{G}^\tau,\mathbb{P}) \times \mathcal{IS}_{BMO}(\mathbb{G}^\tau,\mathbb{P})$. If additionally Assumption 5.4.10 holds, then the bounded solution is unique.*

Proof. Apply Theorem 5.4.18 and Proposition 5.4.19. ☐

In the sequel, we consider a $\mathcal{P}(\mathbb{F})$-measurable process Γ satisfying $|\Gamma|_\infty < +\infty$. Our goal is to show that the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F_\Gamma^{\mathrm{b}}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$ admits a bounded solution. The special case $\Gamma = Y^{\mathrm{d}}_\cdot(\cdot)$ where $(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau))$ is a bounded solution to RBSDE (5.78) will enable us to apply Theorem 5.3.5 to obtain the existence of a bounded solution to RBSDE(F,S,ξ), see Theorem 5.4.24.

Assumption 5.4.6 confers to F^{b} a quadratic growth in its second variable z. However, F^{b} is allowed to have exponential growth in y. Thus we cannot apply directly Theorem 5.4.18 to obtain the existence of a bounded solution to the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F_\Gamma^{\mathrm{b}}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$. To circumvent this difficulty, we will proceed via a classical truncation procedure employed in the setting of BSDEs of quadratic growth in z and possibly exponential growth in y, see [ABSEL10, JKP13, JMPR15]. We have the following result.

Theorem 5.4.22. *Suppose that Assumptions 5.4.3 and 5.4.6 hold. Let Γ be a predictable process satisfying $|\Gamma|_\infty < +\infty$. Then the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^b_\Gamma, S^b, \xi^b)$ admits a bounded solution.*

Proof. For $M > 0$, we consider the function ρ_M and the driver $F^{b,M}_\Gamma$ defined as follows

$$\begin{aligned}\rho_M(y) &= (y \vee -M) \wedge M, y \in \mathbb{R},\\ F^{b,M}_\Gamma(t,\cdot,y,z) &= F^{\mathrm{b}}(t,y,z,\Gamma_t - \rho_M(y)) + \lambda(\Gamma_t - y),\ (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n.\end{aligned}$$

We fix $M > 0$. Using the growth condition (Q1) of Assumption 5.4.6, we have for $(t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n$

$$\begin{aligned}|F^{b,M}_\Gamma(t,y,z)| \ \leq\ & C_1 + C_2 + 2C_y y + 2C_z||z||^2 + \lambda_t|y| + \lambda_t|\Gamma_t\\ &+\lambda_t|\Sigma_t(\Gamma_t - \rho_M(y))| + \lambda_t|J_t(\Gamma_t - \rho_M(y))|.\end{aligned}$$

Since the functions $\Sigma, \lambda J$ are locally Lipschitz, and the processes λ, Γ are bounded, we deduce from the above estimate that there exists $\delta_1, \delta_2 > 0$ depending on M, such that

$$|F^{b,M}_\Gamma(t,y,z)| \ \leq\ \delta_1 + \delta_2|y| + 2C_z||z||^2,\ (t,y,z) \in [0,T] \times \mathbb{R} \times \mathbb{R}^n.$$

The triplet $(F^{b,M}_\Gamma, S^{\mathrm{b}}, \xi^{\mathrm{b}})$ satisfies all the conditions of Theorem 5.4.18. Consequently, there exists a bounded solution $(Y^{b,M}, Z^{b,M}, K^{b,M})$ to the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{b,M}_\Gamma, S^{\mathrm{b}}, \xi^{\mathrm{b}})$. To conclude the proof, it remains to show that we can chose $M > 0$ big enough such that $|Y^{b,M}| \leq M$. For such a value of M, we have $\rho_M(Y^{b,M}) = Y^{b,M}$ and thus $(Y^{b,M}, Z^{b,M}, K^{b,M})$ will define a bounded solution to the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_\Gamma, S^{\mathrm{b}}, \xi^{\mathrm{b}})$. Now as $Y^{b,M}_t \geq S^{\mathrm{b}}_t,\ t \in [0,T]$, and $Y^{b,M}_T = \xi^{\mathrm{b}}$, the process $Y^{b,M}$ admits a lower bound independent of M, namely

$$Y^{b,M}_t \geq -||S^{\mathrm{b}}||_\infty - ||\xi^{\mathrm{b}}||_{L^\infty(\mathcal{F}_T)}. \tag{5.82}$$

Let $M = e^{C_y T}\Big(C_1 T + ||S^{\mathrm{b}}||_\infty + ||\xi^{\mathrm{b}}||_{L^\infty(\mathcal{F}_T)} + |\lambda J|_\infty T + |\Gamma|_\infty\Big)$ and $(Y^{b,M}, Z^{b,M}, K^{b,M})$ be a bounded solution to the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{b,M}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$. We recall that $(Y^{b,M}, Z^{b,M}, K^{b,M})$ satisfies

$$-dY^{b,M}_t \ =\ F^{b,M}(t, Y^{b,M}_t, Z^{b,M}_t, \Gamma_t - \rho_M(Y^{b,M}_t))dt + dK^{b,M}_t - Z^{b,M}_t dB^{\mathbb{F}}_t,\ t \in [0,T].$$

Let $t \in [0,T], \epsilon > 0$ and σ^ϵ be defined as follows:

$$\sigma^\epsilon = \inf\{s \in [t,T] : Y^{b,M}_s \leq S^{\mathrm{b}}_s + \epsilon\} \wedge T.$$

Let β be a real valued $\mathcal{P}(\mathbb{F})$-measurable process to be made precise later. Itô's formula yields

$$\begin{aligned}e^{\int_0^t \beta_r dr} Y^{b,M}_t &= e^{\int_0^{\sigma^\epsilon} \beta_r dr} Y^{b,M}_{\sigma^\epsilon} - \int_t^{\sigma^\epsilon} e^{\int_0^s \beta_r dr} dY^{b,M}_s - \int_t^{\sigma^\epsilon} \beta_s e^{\int_0^s \beta_r dr} Y^{b,M}_s ds, &(5.83)\\ &= e^{\int_0^{\sigma^\epsilon} \beta_r dr} Y^{b,M}_{\sigma^\epsilon} - \int_t^{\sigma^\epsilon} e^{\int_0^s \beta_r dr} Z^{b,M}_s dB^{\mathbb{F}}_s + \int_t^{\sigma^\epsilon} e^{\int_0^s \beta_r dr} dK^{b,M}_s &(5.84)\\ &\quad + \int_t^{\sigma^\epsilon} e^{\int_0^s \beta_r dr}\left[F^{b,M}_\Gamma(s, Y^{b,M}_s, Z^{b,M}_s, \Gamma_s - \rho_M(Y^{b,M}_s)) - \beta_s Y^{b,M}_s\right] ds &(5.85)\end{aligned}$$

The process $\Gamma - \rho_M(Y^{b,M})$ being bounded, the $(\mathbf{A}_\gamma)$ property[6] of λJ implies that there exists a bounded $\mathcal{P}(\mathbb{F})$-measurable process Π^M with $\Pi^M \geq -1$ and such that

$$\lambda J(\Gamma - \rho_M(Y^{b,M})) \ \leq\ \lambda\Pi^M(\Gamma - \rho_M(Y^{b,M})) + \lambda J(0),\ dt \otimes \mathbb{P}\text{-a.e..}$$

[6] See Definition 5.4.4.

Employing (Q1) and the above bound, we obtain for $s \in [0,T]$ the following upper bound for $F^{b,M}_{\Gamma}$

$$
\begin{aligned}
F^{b,M}_{\Gamma}(s, Y^{b,M}_s, Z^{b,M}_s) &\leq C_1 + C_y|Y^{b,M}_s| + C_z|Z^{b,M}_s|^2 + \lambda_s J_s(\Gamma_s - \rho_M(Y^{b,M}_s)) + \lambda_s(\Gamma_s - Y^{b,M}_s),\\
&\leq C_1 + C_y|Y^{b,M}_s| + C_z|Z^{b,M}_s|^2 + \lambda_s J_s(0)\\
&\quad + \lambda_s \Pi^M_s(\Gamma_s - \rho_M(Y^{b,M}_s)) + \lambda_s(\Gamma_s - Y^{b,M}_s).
\end{aligned}
$$

Having chosen $M \geq ||S^{\mathrm{b}}||_\infty + ||\xi^{\mathrm{b}}||_{L^\infty(\mathcal{F}_T)}$, we conclude that (5.82) $Y^{b,M} \geq -M$. Hence $\rho_M(Y^{p,M}) = Y^{b,M} \wedge M$. The facts that $1 + \Pi^M \geq 0$ and $\Gamma - M \leq 0$ lead to the inequality

$$
\lambda_s \Pi^M_s(\Gamma_s - \rho_M(Y^{b,M}_s)) + \lambda_s(\Gamma_s - Y^{b,M}_s) \quad \leq \quad \lambda_s(1 + \Pi^M_s)(\Gamma_s - Y^{b,M}_s)1_{\{Y^{b,M}_s \leq M\}}, \; s \in [0,T].
$$

A combination of the last two estimates yields the following upper bound

$$
\begin{aligned}
F^{b,M}_{\Gamma}(s, Y^{b,M}_s, Z^{b,M}_s) \quad &\leq \quad C_1 + C_z|Z^{b,M}_s|^2 + \lambda_s J_s(0) + \lambda_s(1 + \Pi^M_s)\Gamma_s 1_{\{Y^{b,M}_s \leq M\}} + C_y|Y^{b,M}_s|\\
&\qquad -\lambda_s(1 + \Pi^M_s) Y^{b,M}_s 1_{\{Y^{b,M} \leq M\}}, \; s \in [0,T].
\end{aligned}
$$

One verifies that $|Y^{b,M}_s| = -\mathrm{sign}(-Y^{b,M}_s)Y^{b,M}_s$, $s \in [0,T]$. We now choose β according to

$$
\beta_s \quad = \quad -C_y \mathrm{sign}(-Y^{b,M}_s) - \lambda_s(1 + \Pi^M_s)1_{\{Y^{b,M}_s \leq M\}}, \; s \in [0,T].
$$

As a result of the choice of β, we obtain that for $s \in [0,T]$

$$
F^{b,M}_{\Gamma}(s, Y^{b,M}_s, Z^{b,M}_s) - \beta_s Y^{b,M}_s \leq C_1 + C_z|Z^{b,M}_s|^2 + \lambda_s J_s(0) + \lambda_s(1 + \Pi^M_s)\Gamma_s 1_{\{Y^{b,M}_s \leq M\}}. \tag{5.86}
$$

To obtain an upper bound for $Y^{b,M}_t$, we use the stopping time σ^ϵ. On the set $\{\sigma^\epsilon = T\}, Y_{\sigma^\epsilon} = \xi^{\mathrm{b}}$. The processes $Y^{b,M}$ and S^{b} being continuous, we have $Y^{b,M}_{\sigma^\epsilon} \leq S^{\mathrm{b}}_{\sigma^\epsilon} + \epsilon$ on $\{\sigma^\epsilon < T\}$. Furthermore, we have $Y^{\mathrm{b}}_s > S^{\mathrm{b}}_s, s \in [t, \sigma^\epsilon]$ by definition of σ^ϵ. The Skorohod condition $\int_t^{\sigma^\epsilon}(Y^{b,M}_s - S^{\mathrm{b}}_s)dK^{b,M}_s = 0$ implies that $K^{b,M}_{\sigma^\epsilon} = K^{b,M}_t$. Inserting (5.86) into (5.83) and using the inequality $Y^{b,M}_{\sigma^\epsilon} \leq |\xi^{\mathrm{b}}| + |S^{\mathrm{b}}_{\sigma^\epsilon}| + \epsilon$ yields the bound

$$
\begin{aligned}
Y^{b,M}_t \quad &\leq \quad e^{\int_t^{\sigma^\epsilon}\beta_r dr}\left(|\xi^{\mathrm{b}}| + |S^{\mathrm{b}}_{\sigma^\epsilon}| + \epsilon\right) - \int_t^{\sigma^\epsilon} e^{\int_t^s \beta_r dr}\left(Z^{b,M}_s dB^{\mathbb{F}}_s - C_z|Z^{b,M}_s|^2 ds\right) &(5.87)\\
&\quad + \int_t^{\sigma^\epsilon} e^{\int_t^s \beta_r dr}\left[C_1 + \lambda_s J_s(0) + \lambda_s(1 + \Pi^M_s)\Gamma_s 1_{\{Y^{b,M}_s \leq M\}}\right] ds. &(5.88)
\end{aligned}
$$

Now by Proposition 5.4.19, $Z^{b,M} \in \mathcal{H}^{2,n}_{BMO}(\mathbb{F}, \widehat{\mathbb{Q}})$. It follows from [Kaz94, Theorem 3.6] that the semimartingale $\int_0^\cdot e^{\int_0^s \beta_r dr}\left(Z^{b,M}_s dB^{\mathbb{F}}_s - C_z|Z^{b,M}_s|^2 ds\right)$ is an $(\mathbb{F}, \mathbb{Q}^M)$-martingale where $\mathbb{Q}^M$ is the probability measure with density on $\mathcal{F}_T$ given by

$$
d\mathbb{Q}^M/d\mathbb{P}|_{\mathcal{F}_T} := \mathcal{E}\left(C_z Z^{b,M} \cdot B^{\mathbb{F}}\right)_T.
$$

Clearly, $e^{\int_0^s \beta_r dr} \leq e^{C_y T} e^{-\int_0^s \lambda_r \left(1 + \Pi^M_r\right)1_{\{Y^{b,M}_r \leq M\}}dr}$, $s \in [0,T]$, and by integration

$$
\int_t^{\sigma^\epsilon} e^{\int_t^s \beta_r dr}\lambda_s(1 + \Pi^M_s)1_{\{Y^{b,M}_s \leq M\}} ds \quad \leq \quad e^{C_y T}\left[1 - e^{-\int_t^{\sigma^\epsilon}\lambda_r\left(1 + \Pi^M_r\right)dr}\right] \leq e^{C_y T}.
$$

Taking conditional expectations in (5.87) w.r.t. $\mathcal{F}_t$ under the measure $\mathbb{Q}^M$ together with the latter inequalities, we obtain

$$
Y^{b,M}_t \leq e^{C_y T}\left[|\xi^{\mathrm{b}}|_{L^\infty(\mathcal{F}_T)} + ||S^{\mathrm{b}}||_\infty + C_1 T + |\lambda J(0)|_\infty + |\Gamma|_\infty\right] + \epsilon.
$$

Taking the limit as ϵ goes to 0, we see that $|Y^{b,M}| \leq M$. Thus $(Y^{b,M}, Z^{b,M}, K^{b,M})$ is a bounded solution to the $\mathrm{RBSDE}_{(\mathbb{F}, \widehat{\mathbb{Q}})}(F^{\mathrm{b}}_{\Gamma}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$. □

Remark 5.4.23. *For a pair of data (F^b, S^b, ξ^b) and $(\overline{F}^b, \overline{S}^b, \overline{\xi}^b)$ satisfying Assumptions 5.4.3 and 5.4.10, a priori estimates for the difference of solutions to the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^b_\Gamma, S^b, \xi^b)$ and RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(\overline{F}^b_\Gamma, \overline{S}^b, \overline{\xi}^b)$ similar to those in Proposition 5.4.15 can be established using analogous arguments. One can therefore deduce w.l.o.g. that the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^b_\Gamma, S^b, \xi^b)$ admits a unique bounded solution under Assumption 5.4.10.*

Combining together Theorems 5.4.21 and 5.4.22, we obtain the following existence result for the RBSDE(F, S, ξ).

Theorem 5.4.24. *Suppose that Assumptions 5.4.3 and 5.4.6 hold. Then the RBSDE(F, S, ξ) admits a bounded solution (Y, Z, U, K). If Assumption 5.4.10 holds instead of Assumption 5.4.6, then there exists a unique bounded solution.*

Proof. By Theorem 5.4.21, the RBSDE$_{(\mathbb{G}^\tau,\mathbb{P})}(F^{\mathrm{d}}, S^{\mathrm{d}}(\tau), \xi^{\mathrm{d}}(\tau))$ admits a bounded solution which we denote by $\big(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau)\big)$. Since $\mathbb{F}$ is continuous, $Y^{\mathrm{d}}_\cdot(\cdot)$ is $\mathcal{P}(\mathbb{F})$-measurable. Moreover, $|Y^{\mathrm{d}}_\cdot(\cdot)|_\infty < +\infty$. We infer from Theorem 5.4.22 that the RBSDE$_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_{Y^{\mathrm{d}}_\cdot(\cdot)}, S^{\mathrm{d}}, \xi^{\mathrm{b}})$ admits a bounded solution $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, K^{\mathrm{b}})$. Theorem 5.3.5 implies that (Y, Z, U, K) defined as in (5.28) is a solution to the RBSDE(F, S, ξ). Since Y^{b} and $Y^{\mathrm{d}}(\tau)$ are bounded, Y is bounded. Thus (Y, Z, U, K) is a bounded solution. If Assumption 5.4.10 holds instead of Assumption 5.4.6, then uniqueness of a bounded solution follows from Corollary 5.4.16. $\square$

5.4.3 Comparison principle

Having established existence and uniqueness of solutions, we will in this section establish a comparison principle for RBSDEs (5.50) with data satisfying Assumptions 5.4.3 and 5.4.10. Similarly as for the existence result, our comparison principle will be based on the two step methodology developed above. This will allow us to avoid the strong $(\mathbf{A}_\gamma)$ condition often employed in the literature for BSDEs with jumps [Roy06, PKTZ15, Mor09b, Ngo10].

Theorem 5.4.25. *Let $(\overline{F}, \overline{S}, \overline{\xi})$ be another set of data with optional splitting formula*

$$\overline{F}(t,\cdot,y,z,u) = \overline{F}^b(t,\cdot,y,z,u)1_{\{t\le\tau\}} + \overline{F}^d(t,\cdot,y,z)1_{\{t>\tau\}},\ (t,y,z,u)\in[0,T]\times\mathbb{R}^{n+2}, \tag{5.89}$$

$$\overline{S} = \overline{S}^b_t 1_{\{t<\tau\}} + \overline{S}^d_t(\tau)1_{\{t\ge\tau\}},\ t\in[0,T], \tag{5.90}$$

$$\overline{\xi} = \overline{\xi}^b 1_{\{T<\tau\}} + \overline{\xi}^d(\tau)1_{\{T\ge\tau\}}. \tag{5.91}$$

Suppose that Assumptions 5.4.3 and 5.4.10 hold for the pairs (F, S, ξ) and $(\overline{F}, \overline{S}, \overline{\xi})$. Let (Y, Z, U, K) (resp. $(\overline{Y}, \overline{Z}, \overline{U}, \overline{K})$) be the unique bounded solution to the RBSDE(F, S, ξ) (resp. RBSDE$(\overline{F}, \overline{S}, \overline{\xi})$). If

a) $F(\cdot, \overline{Y}, \overline{Z}, \overline{U}) \le \overline{F}(\cdot, \overline{Y}, \overline{Z}, \overline{U})$, $dt\otimes\mathbb{P}$-a.e..,

b) $S_t \le \overline{S}_t$, $t\in[0,T]$,

c) $\xi \le \overline{\xi}$,

then $Y_t \le \overline{Y}_t$ for all $t\in[0,T]$.

In the sequel, for $x\in\mathbb{R}$, $x^+ = \max\{x, 0\}$.

Proof. By Theorems 5.3.5 and 5.4.24, we know that for $t\in[0,T]$, the solutions (Y_t, Z_t, U_t, K_t) and $(\overline{Y}_t, \overline{Z}_t, \overline{U}_t, \overline{K}_t)$ are of the form

$$\left\{\begin{array}{ll} Y_t & = Y^{\mathrm{b}}_t 1_{\{t<\tau\}} + Y^{\mathrm{d}}_t(\tau)1_{\{t\ge\tau\}},\\ Z_t & = Z^{\mathrm{b}}_t 1_{\{t\le\tau\}} + Z^{\mathrm{d}}_t(\tau)1_{\{t>\tau\}},\\ U_t & = (Y^{\mathrm{d}}_t(t) - Y^{\mathrm{b}}_{t-})1_{\{t\le\tau\}},\\ K_t & = K^{\mathrm{b}}_{t\wedge\tau} + (K^{\mathrm{d}}_t(\tau) - K^{\mathrm{d}}_\tau(\tau))1_{\{t>\tau\}}, \end{array}\right. \quad \text{and} \quad \left\{\begin{array}{ll} \overline{Y}_t & = \overline{Y}^{\mathrm{b}}_t 1_{\{t<\tau\}} + \overline{Y}^{\mathrm{d}}_t(\tau)1_{\{t\ge\tau\}},\\ \overline{Z}_t & = \overline{Z}^{\mathrm{b}}_t 1_{\{t\le\tau\}} + \overline{Z}^{\mathrm{d}}_t(\tau)1_{\{t>\tau\}},\\ \overline{U}_t & = (\overline{Y}^{\mathrm{d}}_t(t) - \overline{Y}^{\mathrm{b}}_{t-})1_{\{t\le\tau\}},\\ \overline{K}_t & = \overline{K}^{\mathrm{b}}_{t\wedge\tau} + (\overline{K}^{\mathrm{d}}_t(\tau) - \overline{K}^{\mathrm{d}}_\tau(\tau))1_{\{t>\tau\}}, \end{array}\right. \tag{5.92}$$

where the triplet $(Y^{\mathrm{d}}(\tau), Z^{\mathrm{d}}(\tau), K^{\mathrm{d}}(\tau))$ (resp. $(\overline{Y}^{\mathrm{d}}(\tau), \overline{Z}^{\mathrm{d}}(\tau), \overline{K}^{\mathrm{d}}(\tau)))$ is the unique bounded solution to the $\mathrm{RBSDE}_{(\mathbb{G}^\tau,\mathbb{P})}(F^{\mathrm{d}}, S^{\mathrm{d}}(\tau), \xi^{\mathrm{d}}(\tau))$ (resp. $\mathrm{RBSDE}_{(\mathbb{G}^\tau,\mathbb{P})}(\overline{F}^{\mathrm{d}}, \overline{S}^{\mathrm{d}}(\tau), \overline{\xi}^{\mathrm{d}}(\tau)))$ and the triplet $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, K^{\mathrm{b}})$ (resp. $(\overline{Y}^{\mathrm{b}}, \overline{Z}^{\mathrm{b}}, \overline{K}^{\mathrm{b}}))$ is the unique bounded solution to the pre-default $\mathrm{RBSDE}_{(\mathbb{F},\widehat{\mathbb{Q}})}(F^{\mathrm{b}}_{Y^{\mathrm{d}}_{\cdot}(\cdot)}, S^{\mathrm{b}}, \xi^{\mathrm{b}})$ (resp. $\mathrm{RBSDE}_{(\mathbb{F},\widehat{\mathbb{Q}})}(\overline{F}^{\mathrm{b}}_{\overline{Y}^{\mathrm{d}}_{\cdot}(\cdot)}, \overline{S}^{\mathrm{b}}, \overline{\xi}^{\mathrm{b}})$. To prove the result, we will proceed in two steps:

Step 1. We show that for $t \in [0, T]$ we have $Y^{\mathrm{d}}_t(\tau)1_{\{t\geq\tau\}} \leq \overline{Y}^{\mathrm{d}}_t(\tau)1_{\{t\geq\tau\}}$. Let P, Q, V and f be the processes defined for $t \in [0, T]$ by

$$\begin{aligned} P_t &= Y^{\mathrm{d}}_t(\tau) - \overline{Y}^{\mathrm{d}}_t(\tau),\ Q_t = Z^{\mathrm{d}}_t(\tau) - \overline{Z}^{\mathrm{d}}_t(\tau),\ V_t = K^{\mathrm{d}}_t(\tau) - \overline{K}^{\mathrm{d}}_t(\tau) \text{ and} \\ f_t &= -F^{\mathrm{d}}(t, \overline{Y}^{\mathrm{d}}_t(\tau), \overline{Z}^{\mathrm{d}}_t(\tau)) + \overline{F}^{\mathrm{d}}(t, \overline{Y}^{\mathrm{d}}_t(\tau), \overline{Z}^{\mathrm{d}}_t(\tau)). \end{aligned}$$

Using the equations describing the dynamics for $Y^{\mathrm{d}}(\tau)$ and $\overline{Y}^{\mathrm{d}}(\tau)$, we have for $t \in [0, T]$

$$\begin{aligned} dP_t &= \Big(-F^{\mathrm{d}}(t, Y^{\mathrm{d}}_t(\tau), Z^{\mathrm{d}}_t(\tau)) + \overline{F}^{\mathrm{d}}(t, \overline{Y}^{\mathrm{d}}_t(\tau), \overline{Z}^{\mathrm{d}}_t(\tau))\Big)\, dt - dV_t + Q_t dB^{\mathbb{G}^\tau}_t \\ &= \Big(-\varphi^{\mathrm{d}}_t(\tau)P_t - Q_t\psi^{\mathrm{d}}_t(\tau) + f_t\Big)\, dt - dV_t + Q_t dB^{\mathbb{G}^\tau}_t, \end{aligned}$$

where $\varphi^{\mathrm{d}}(\tau)$ and $\psi^{\mathrm{d}}(\tau)$ are defined for $t \in [0, T]$ by

$$\begin{aligned} \varphi^{\mathrm{d}}_t(\tau) &= \frac{F^{\mathrm{d}}(t, Y^{\mathrm{d}}_t(\tau), Z^{\mathrm{d}}_t(\tau)) - F^{\mathrm{d}}(t, \overline{Y}^{\mathrm{d}}_t(\tau), Z^{\mathrm{d}}_t(\tau))}{Y^{\mathrm{d}}_t(\tau) - \overline{Y}^{\mathrm{d}}_t(\tau)} 1_{\{Y^{\mathrm{d}}_t(\tau)\neq\overline{Y}^{\mathrm{d}}_t(\tau)\}}, \\ \psi^{\mathrm{d}}_t(\tau) &= \frac{F^{\mathrm{d}}(t, \overline{Y}^{\mathrm{d}}_t(\tau), Z^{\mathrm{d}}_t(\tau)) - F^{\mathrm{d}}(t, \overline{Y}^{\mathrm{d}}_t(\tau), \overline{Z}^{\mathrm{d}}_t(\tau))}{||Z^{\mathrm{d}}_t(\tau) - \overline{Z}^{\mathrm{d}}_t(\tau)||^2} (Z^{\mathrm{d}}_t(\tau) - \overline{Z}^{\mathrm{d}}_t(\tau)) 1_{\{||Z^{\mathrm{d}}_t(\tau)-\overline{Z}^{\mathrm{d}}_t(\tau)||\neq 0\}}. \end{aligned}$$

Let L^P denote the local time of P at 0. Then by Itô-Tanaka's formula, we have for $t \in [0, T]$

$$\begin{aligned} dP^+_t &= 1_{\{P_s>0\}} dP_t + \frac{1}{2} dL^P_t, \\ &= 1_{\{P_s>0\}} \Big(-\varphi^{\mathrm{d}}_t(\tau)P_t - Q_t\psi^{\mathrm{d}}_t(\tau) + f_t\Big)\, dt - 1_{\{P_t>0\}} dV_t + 1_{\{P_t>0\}} Q_t dB^{\mathbb{G}^\tau}_t + \frac{1}{2} dL^P_t. \end{aligned}$$

Note that $\Big[1_{\{P_s>0\}}\varphi^{\mathrm{d}}_s(\tau)P_s - \varphi^{\mathrm{d}}_s(\tau)P^+_s\Big] = 0$. Hence Itô's formula and the above equation for P^+ yield

$$e^{\int_0^T \varphi^{\mathrm{d}}_r(\tau)dr} P^+_t = e^{\int_0^T \varphi^{\mathrm{d}}_r(\tau)dr} P^+_T - \int_t^T e^{\int_0^s \varphi^{\mathrm{d}}_r(\tau)dr} dP^+_s - \int_t^T \varphi^{\mathrm{d}}_s(\tau) e^{\int_0^s \varphi^{\mathrm{d}}_r(\tau)dr} P^+_s ds \tag{5.93}$$

$$= e^{\int_0^T \varphi^{\mathrm{d}}_r(\tau)dr} P^+_T - \frac{1}{2}\int_t^T e^{\int_0^s \varphi^{\mathrm{d}}_r(\tau)dr} dL^P_s + \int_t^T e^{\int_0^s \varphi^{\mathrm{d}}_r(\tau)dr} 1_{\{P_s>0\}} dV_s \tag{5.94}$$

$$- \int_t^T e^{\int_0^s \varphi^{\mathrm{d}}_r(\tau)dr} f_s ds + \int_t^T e^{\int_0^s \varphi^{\mathrm{d}}_r(\tau)dr} 1_{\{P_s>0\}} Q_s(\psi^{\mathrm{d}}_s(\tau)ds - dB^{\mathbb{G}^\tau}_s) \tag{5.95}$$

We now fix $t \in [0, T]$. The expression in (5.94) is negative on the event $\{t \geq \tau\}$. Indeed, the inclusion $\{t \geq \tau\} \subseteq \{T \geq \tau\}$ and hypothesis c) lead to the inequality

$$P^+_T 1_{\{t\geq\tau\}} \leq P^+_T 1_{\{T\geq\tau\}} = (\xi^{\mathrm{d}}(\tau)1_{\{T\geq\tau\}} - \overline{\xi}^{\mathrm{d}}(\tau)1_{\{T\geq\tau\}})^+ \leq 0. \tag{5.96}$$

From hypothesis b), we obtain that $S^{\mathrm{d}}_s(\tau)1_{\{s\geq\tau\}} \leq \overline{S}^{\mathrm{d}}_s(\tau)1_{\{s\geq\tau\}}, s \in [0, T]$. Therefore on the random set $\{P > 0\} \cap ([t, T] \times \{t \geq \tau\})$ we have

$$Y^{\mathrm{d}}(\tau) > \overline{Y}^{\mathrm{d}}(\tau) \geq \overline{S}^{\mathrm{d}}(\tau) \geq S^{\mathrm{d}}(\tau). \tag{5.97}$$

The condition $\int_0^T (Y_s^{\mathrm{d}}(\tau) - S_s^{\mathrm{d}}(\tau))dK_s^{\mathrm{d}}(\tau) = 0$ and (5.97) guarantee that $dK^{\mathrm{d}}(\tau) = 0$ on the random set $\{P > 0\} \cap ([t,T] \times \{t \geq \tau\})$. The process $\overline{K}^{\mathrm{d}}(\tau)$ being increasing, we deduce that

$$1_{\{t\geq\tau\}} \int_t^T e^{\int_0^s \varphi_r^{\mathrm{d}}(\tau)dr} 1_{\{P_s>0\}} dV_s = 1_{\{t\geq\tau\}} \int_t^T e^{\int_0^s \varphi_r^{\mathrm{d}}(\tau)dr} 1_{\{P_s>0\}} (dK_s^{\mathrm{d}}(\tau) - d\overline{K}_s^{\mathrm{d}}(\tau)) \leq 0. \quad (5.98)$$

We recall that L^P is an increasing process. The negativity of the expression in (5.94) on the event $\{t \geq \tau\}$ is a consequence of (5.96) and (5.98). Using the optional splitting formulas for $F, \overline{F}$ and (5.92), hypothesis a) gives

$$f_s(\omega) \geq 0 \ \text{ for } ds \otimes \mathbb{P} - a.e\ (s,\omega) \in [t,T] \times \{t \geq \tau\}. \quad (5.99)$$

We deduce from (5.93) that

$$e^{\int_0^t \varphi_r^{\mathrm{d}}(\tau)dr} P_t^+ \times 1_{\{t\geq\tau\}} \quad \leq \quad \int_t^T e^{\int_0^s \varphi_r^{\mathrm{d}}(\tau)dr} 1_{\{P_s>0\}} Q_s(\psi_s^{\mathrm{d}}(\tau)ds - dB_s^{\mathbb{G}^\tau}) \times 1_{\{t\geq\tau\}}. \quad (5.100)$$

By Proposition 5.4.19, $Z^{\mathrm{d}}(\tau), \overline{Z}^{\mathrm{d}}(\tau) \in \mathcal{H}^{2,n}_{BMO}(\mathbb{G}^\tau, \mathbb{P})$. Assumption 5.4.10 entails that $|\varphi^{\mathrm{d}}(\tau)| \leq C_y$ and $|\psi^{\mathrm{d}}(\tau)| \leq C_z(1 + |Z^{\mathrm{d}}(\tau)| + |\overline{Z}^{\mathrm{d}}(\tau)|)$. Thus $Q, \psi^{\mathrm{d}}(\tau) \in \mathcal{H}^{2,n}_{BMO}(\mathbb{G}^\tau, \mathbb{P})$ and by [Kaz94, Theorem 3.6] $Q \cdot \widehat{B}^{\mathbb{G}^\tau}$ is a $(\mathbb{G}^\tau, \mathbb{Q})$-BMO martingale where $d\widehat{B}_s^{\mathbb{G}^\tau} = dB_s^{\mathbb{G}^\tau} - \psi_s^{\mathrm{d}}(\tau)ds, s \in [0,T]$ and $\mathbb{Q}$ is the probability measure equivalent to $\mathbb{P}$ with density on $\mathcal{G}_T^\tau$ given by

$$d\mathbb{Q}/d\mathbb{P} = \mathcal{E}\left(\psi^{\mathrm{d}}(\tau) \cdot B^{\mathbb{G}^\tau}\right)_T.$$

Noting that $1_{\{t\geq\tau\}}$ is $\mathcal{G}_t^\tau$-measurable and $e^{\int_0^\cdot \varphi_r^{\mathrm{d}}(\tau)dr}$ is bounded, taking conditional expectations in (5.100) w.r.t. $\mathcal{G}_t^\tau$ under the measure $\mathbb{Q}$, we see that $e^{\int_0^t \varphi_r^{\mathrm{d}}(\tau)dr} P_t^+ \times 1_{\{t\geq\tau\}} \leq 0$. The latter inequality implies that $Y_t^{\mathrm{d}}(\tau)1_{\{t\geq\tau\}} \leq \overline{Y}_t^{\mathrm{d}}(\tau)1_{\{t\geq\tau\}}$. This proves **Step 1**.

Step 2 We show that $Y_t^{\mathrm{b}} \leq \overline{Y}_t^{\mathrm{b}}, t \in [0,T]$. We will proceed using similar arguments as in **Step 1** with the appropriate equations for Y^{b} and $\overline{Y}^{\mathrm{b}}$. Before, note that taking $t = T \wedge \tau$, the same arguments as in **Step 1** lead to the inequality

$$Y_\tau^{\mathrm{d}}(\tau)1_{\{T\geq\tau\}} \leq \overline{Y}_\tau^{\mathrm{d}}(\tau)1_{\{T\geq\tau\}}.$$

The latter inequality together with Lemma 4.2.16 imply that

$$Y_\cdot^{\mathrm{d}}(\cdot) \leq \overline{Y}_\cdot^{\mathrm{d}}(\cdot),\ dt \otimes \mathbb{P}\text{-a.e.}. \quad (5.101)$$

We introduce the processes $\overline{P}, \overline{Q}, \overline{V}$ and $\overline{f}$ and Ξ where for $t \in [0,T]$

$$\begin{aligned}
\overline{P}_t &= Y_t^{\mathrm{b}} - \overline{Y}_t^{\mathrm{b}},\ \overline{Q}_t = Z_t^{\mathrm{b}} - \overline{Z}_t^{\mathrm{b}},\ \overline{V}_t = K_t^{\mathrm{b}} - \overline{K}_t^{\mathrm{b}}, \\
\overline{f}_t &= -F^{\mathrm{b}}(t, \overline{Y}_t^{\mathrm{b}}, \overline{Z}_t^{\mathrm{b}}, \overline{Y}_t^{\mathrm{d}}(t) - \overline{Y}_t^{\mathrm{b}}) + \overline{F}^{\mathrm{b}}(t, \overline{Y}_t^{\mathrm{b}}, \overline{Z}_t^{\mathrm{b}}, \overline{Y}_t^{\mathrm{d}}(t) - \overline{Y}_t^{\mathrm{b}}), \\
\Xi_t &= -F^{\mathrm{b}}(t, \overline{Y}_t^{\mathrm{b}}, \overline{Z}_t^{\mathrm{b}}, Y_t^{\mathrm{d}}(t) - Y_t^{\mathrm{b}}) + F^{\mathrm{b}}(t, \overline{Y}_t^{\mathrm{b}}, \overline{Z}_t^{\mathrm{b}}, \overline{Y}_t^{\mathrm{d}}(t) - \overline{Y}_t^{\mathrm{b}}), \\
\varphi_t^{\mathrm{b}} &= \frac{F^{\mathrm{b}}(t, Y_t^{\mathrm{b}}, Z_t^{\mathrm{b}}, Y_t^{\mathrm{d}}(t) - Y_t^{\mathrm{b}}) - F^{\mathrm{b}}(t, \overline{Y}_t^{\mathrm{b}}, Z_t^{\mathrm{b}}, Y_t^{\mathrm{d}}(t) - Y_t^{\mathrm{b}})}{\overline{P}_t} 1_{\{\overline{P}_t \neq 0\}}, \\
\psi_t^{\mathrm{b}} &= \frac{F^{\mathrm{b}}(t, \overline{Y}_t^{\mathrm{b}}, Z_t^{\mathrm{b}}, Y_t^{\mathrm{d}}(t) - Y_t^{\mathrm{b}}) - F^{\mathrm{b}}(s, \overline{Y}_t^{\mathrm{b}}, \overline{Z}_t^{\mathrm{b}}, Y_t^{\mathrm{d}}(t) - Y_t^{\mathrm{b}})}{||\overline{Q}_t||^2} \overline{Q}_t 1_{\{||\overline{Q}_t||\neq 0\}}.
\end{aligned}$$

Recall that the triplets $(Y^{\mathrm{b}}, Z^{\mathrm{b}}, K^{\mathrm{b}})$ and $(\overline{Y}^{\mathrm{b}}, \overline{Z}^{\mathrm{d}}, \overline{K}^{\mathrm{b}})$ satisfy for $t \in [0,T]$

$$\begin{aligned}
dY_t^{\mathrm{b}} &= F^{\mathrm{b}}(t, Y_t^{\mathrm{b}}, Z_t^{\mathrm{b}}, Y_t^{\mathrm{d}}(t) - Y_t^{\mathrm{b}})dt - \lambda_t(Y_t^{\mathrm{d}}(t) - Y_t^{\mathrm{b}})dt - dK_t^{\mathrm{b}} - Z_t^{\mathrm{b}} dB_t^{\mathbb{F}} \\
d\overline{Y}_t^{\mathrm{b}} &= \overline{F}^{\mathrm{b}}(t, \overline{Y}_t^{\mathrm{b}}, \overline{Z}_t^{\mathrm{b}}, \overline{Y}_t^{\mathrm{d}}(t) - \overline{Y}_t^{\mathrm{b}})dt - \lambda_t(\overline{Y}_t^{\mathrm{d}}(t) - \overline{Y}_t^{\mathrm{b}})dt - d\overline{K}_t^{\mathrm{b}} + \overline{Z}_t^{\mathrm{b}} dB_t^{\mathbb{F}}.
\end{aligned}$$

Using the processes $\overline{Q}, \overline{V}, \overline{f}, \Xi, \varphi^{\mathrm{b}}$ and ψ^{b}, one sees that the dynamics of $\overline{P}$ is given for $s \in [0,T]$ by

$$d\overline{P}_s = \Big((\lambda_s - \varphi^{\mathrm{b}}_s)\overline{P}_s + \lambda_s(\overline{Y}^{\mathrm{d}}_s(s) - Y^{\mathrm{d}}_s(s)) + \Xi_s + \overline{f}_s \Big)\, ds - d\overline{V}_s + \overline{Q}_s\big(- \psi_s ds + dB^{\mathbb{F}}_s\big). \quad (5.102)$$

Let $L^{\overline{P}}$ be the local time of $\overline{P}$ at 0. Then Itô-Tanaka's formula and (5.102) give for $t \in [0,T]$

$$\begin{aligned} d(\overline{P})^+_t &= 1_{\{\overline{P}_t>0\}} d\overline{P}_s + \frac{1}{2} dL^{\overline{P}}_t \\ &= 1_{\{\overline{P}_t>0\}} [\overline{f}_t dt - d\overline{V}_t] + \frac{1}{2} dL^{\overline{P}}_t + 1_{\{\overline{P}_t>0\}} \overline{Q}_t\big(- \psi^{\mathrm{b}}_t dt + dB^{\mathbb{F}}_t\big) \\ &\quad + 1_{\{\overline{P}_t>0\}} \Big((- \varphi^{\mathrm{b}}_t + \lambda_t)\overline{P}_t + \Xi_t + \lambda_t(\overline{Y}^{\mathrm{d}}_t(t) - Y^{\mathrm{d}}_t(t)) \Big)\, dt. \end{aligned}$$

Let $t \in [0,T]$ and β be a real valued $\mathcal{P}(\mathbb{F})$-measurable process to be chosen later. Itô's formula yields

$$\begin{aligned} e^{\int_0^t \beta_r dr} (\overline{P})^+_t &= e^{\int_0^T \beta_r dr} (\overline{P})^+_T - \int_t^T e^{\int_0^s \beta_r dr} (\overline{P})^+_s \beta_s ds - \int_t^T e^{\int_0^s \beta_r dr} d(\overline{P})^+_s \quad (5.103) \\ &= e^{\int_0^T \beta_r dr} (\overline{P})^+_T - \frac{1}{2} \int_t^T e^{\int_0^s \beta_r dr} dL^{\overline{P}}_s + \int_t^t e^{\int_0^s \beta_r dr} 1_{\{\overline{P}_s>0\}} (d\overline{V}_s - \overline{f}_s ds) \\ &\quad \int_t^T e^{\int_0^s \beta_r dr} \Big[-\beta_s(\overline{P})^+_s + 1_{\{\overline{P}_s>0\}} \Big((\varphi^{\mathrm{b}}_s - \lambda_s)\overline{P}_s - \Xi_s + \lambda_s(Y^{\mathrm{d}}_s(s) - \overline{Y}^{\mathrm{d}}_s(s)\Big) \Big] ds \\ &\quad \int_t^T e^{\int_0^s \beta_r dr} 1_{\{\overline{P}_s>0\}} \overline{Q}_s(\psi^{\mathrm{b}}_s ds - dB^{\mathbb{F}}_s). \end{aligned}$$

To conclude the proof, it remains to show that by conditioning the above equation w.r.t. $\mathcal{F}_t$ under a suitable measure, the terms appearing on the right hand side are negative. We recall that $G_s = \mathbb{P}\left(\tau > s | \mathcal{F}_s\right),\ s \in [0,T]$. By hypothesis c) and the density hypothesis, we have

$$\xi^{\mathrm{b}} 1_{\{T<\tau\}} \leq \overline{\xi}^{\mathrm{b}} 1_{\{T<\tau\}} \Rightarrow \xi^{\mathrm{b}} G_T \leq \overline{\xi}^{\mathrm{b}} G_T \Rightarrow \xi^{\mathrm{b}} \leq \overline{\xi}^{\mathrm{b}}. \quad (5.104)$$

As a result, $(\overline{P})^+_T = 0$. Observe that from the hypothesis b), $S^{\mathrm{b}}_s 1_{\{s<\tau\}} \leq \overline{S}^{\mathrm{b}} 1_{\{s<\tau\}},\ s \in [0,T]$. Thus for each $s \in [0,T]$, conditioning the latter inequality w.r.t. $\mathcal{F}_s$ and dividing by G_s one sees that

$$S^{\mathrm{b}}_s \leq \overline{S}^{\mathrm{b}}_s,\ s \in [0,T].$$

Noting that on $\{\overline{P} > 0\}$, we have $Y^{\mathrm{b}} > \overline{Y}^{\mathrm{b}} \geq \overline{S}^{\mathrm{b}} \geq S^{\mathrm{b}}$. The Skorohod condition $\int_0^T (Y^{\mathrm{b}}_s - S^{\mathrm{b}}_s) dK^{\mathrm{b}}_s = 0$ implies that $dK^{\mathrm{b}} = 0$ on $\{\overline{P} > 0\}$. We infer from the monotonicity property of $\overline{K}^{\mathrm{b}}$ that

$$\int_t^T e^{\int_0^s \beta_r dr} 1_{\{\overline{P}_s>0\}} d\overline{V}_s = \int_t^T e^{\int_0^s \beta_r dr} 1_{\{\overline{P}_s>0\}} (dK^{\mathrm{b}}_s - d\overline{K}^{\mathrm{b}}_s) \leq 0. \quad (5.105)$$

The optional splitting formulas of F, S, the decomposition (5.92) and hypothesis a) ensure that for Lebesgue-almost all $s \in [0,T]$

$$-\overline{f}_s 1_{\{s<\tau\}} = \Big(F^{\mathrm{b}}(s, \overline{Y}^{\mathrm{b}}_s, \overline{Z}^{\mathrm{b}}_s, \overline{Y}^{\mathrm{d}}_s(s) - \overline{Y}^{\mathrm{b}}_s) - \overline{F}^{\mathrm{b}}(s, \overline{Y}^{\mathrm{b}}_s, \overline{Z}^{\mathrm{b}}_s, \overline{Y}^{\mathrm{d}}_s(s) - \overline{Y}^{\mathrm{b}}_s) \Big) 1_{\{s<\tau\}},\ \mathbb{P}\text{-a.s.} \quad (5.106)$$

Taking conditional expectations in (5.106) w.r.t. $\mathcal{F}_s$ for Lebesgue-almost all $s \in [0,T]$ and dividing by G_s, we deduce that

$$\overline{f}_s \geq 0, ds \otimes \mathbb{P}\text{-a.e.}. \quad (5.107)$$

As F^{b} satisfies the $(\mathbf{A}_\gamma)$ condition, there exists a real valued bounded $\mathcal{P}(\mathbb{F})$-measurable function Π with $\Pi \geq -1$ such that for $s \in [0,T]$

$$-\Xi_s = F^{\mathrm{b}}(s, \overline{Y}^{\mathrm{b}}_s, \overline{Z}^{\mathrm{b}}_s, Y^{\mathrm{d}}_s(s) - Y^{\mathrm{b}}_s) - F^{\mathrm{b}}(s, \overline{Y}^{\mathrm{b}}_s, \overline{Z}^{\mathrm{b}}_s, \overline{Y}^{\mathrm{d}}_s(s) - \overline{Y}^{\mathrm{b}}_s) \leq \lambda_s \Pi_s (Y^{\mathrm{d}}_s(s) - \overline{Y}^{\mathrm{d}}_s(s) - \overline{P}_s).$$

We now choose $\beta_s = \varphi_s - \lambda_s(1+\Pi_s), s \in [0,T]$. As $1+\Pi \geq 0$ and $Y^{\mathrm{d}}_\cdot(\cdot) \leq \overline{Y}^{\mathrm{d}}_\cdot(\cdot)$ by (5.101), we deduce from the above estimate that for $s \in [0,T]$

$$-\beta_s(\overline{P})^+_s + 1_{\{\overline{P}_s>0\}}\Big((\varphi^{\mathrm{b}}_s - \lambda_s)\overline{P}_s - \Xi_s + \lambda_s(Y^{\mathrm{d}}_s(s) - \overline{Y}^{\mathrm{d}}_s(s)\Big) \leq 0. \tag{5.108}$$

Inserting inequalities (5.104), (5.105), (5.107) and (5.108) into (5.103), we obtain

$$e^{\int_0^t \beta_r dr}(\overline{P})^+_t \leq \int_t^T e^{\int_0^s \beta_r dr} 1_{\{\overline{P}_s>0\}} \overline{Q}_s (\psi^{\mathrm{b}}_s ds - dB^{\mathbb{F}}_s). \tag{5.109}$$

By Assumption 5.4.10, $|\varphi^{\mathrm{b}}| \leq C_y$ and $|\psi^{\mathrm{b}}| \leq C_z(1+|Z^{\mathrm{b}}|+|\overline{Z}^{\mathrm{b}}|)$. Since $Z^{\mathrm{b}}, \overline{Z}^{\mathrm{b}} \in \mathcal{H}^{2,n}_{BMO}(\mathbb{F}, \widehat{\mathbb{Q}})$, we have $\psi^{\mathrm{b}} \in \mathcal{H}^{2,n}_{BMO}(\mathbb{F}, \widehat{\mathbb{Q}})$. We consider the measure $\overline{\mathbb{Q}}$ with density on $\mathcal{F}_T$ given by

$$d\overline{\mathbb{Q}}/d\mathbb{P}|_{\mathcal{F}_T} := \mathcal{E}(\psi^{\mathrm{b}} \cdot B^{\mathbb{F}}).$$

We infer from [Kaz94, Theorem 3.6] that $\overline{Q} \cdot \widehat{B}^{\mathbb{F}}$ is an $(\mathbb{F}, \overline{\mathbb{Q}})$-BMO martingale. The process β being bounded, taking conditional expectations in (5.109) w.r.t. $\mathcal{F}_t$ under the measure $\overline{\mathbb{Q}}$ yields the inequality $(\overline{P})^+_t \leq 0$. Hence $Y^{\mathrm{b}}_t \leq \overline{Y}^{\mathrm{b}}_t$. **Step 2** is complete.

Combining **Step 1** and **Step 2**, we see that the inequality $Y_t \leq \overline{Y}_t$ is a consequence of the splitting formula (5.92). $\square$

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Bibliography

[ABSEL10] S. Ankirchner, C. Blanchet-Scalliet, and A. Eyraud-Loisel. Credit risk premia and quadratic BSDEs with a single jump. *International Journal of Theoretical and Applied Finance (IJTAF)*, 13(07):1103–1129, 2010.

[AH06] F. Antonelli and S. Hamadène. Existence of the solutions of backward–forward SDE's with continuous monotone coefficients. *Statistics & Probability Letters*, 76(14):1559–1569, 2006.

[AIDR07] S. Ankirchner, P. Imkeller, and G. Dos Reis. Classical and varitional differentiability of BSDEs with quadratic growth. *Electronic Journal of Probability*, 12(53):1418–1453, 2007.

[AIS98] J. Amendinger, P. Imkeller, and M. Schweizer. Additional logarithmic utility of an insider. *Stochastic Processes and their Applications*, 75(2):263–286, 1998.

[Ama09] A. Aman. L^p-solution of reflected generalized BSDEs with non-Lipschitz coefficients. *Random Operators and Stochastic Equations*, 17(3):201–219, 2009.

[Ame00] J. Amendinger. Martingale representation theorems for initially enlarged filtrations. *Stochastic Processes and their Applications*, 89(1):101–116, 2000.

[Ank05] S. Ankirchner. *Information and semimartingales*. PhD thesis, Humboldt Universität zu Berlin, 2005.

[Arr65] K. J. Arrow. *Aspects of the theory of risk-bearing*. Yrjö Jahnsson lectures. Helsinki, Yrjö Jahnssonin Säätiö, 1965.

[AS92] J.-P. Ansel and C. Stricker. Lois de martingale, densités et décomposition de Föllmer-Schweizer. *Annales de l'Institiut Henri Poincaré*, 28(3):375–392, 1992.

[AS93] J.-P. Ansel and C. Stricker. Décomposition de Kunita-Watanabe. In *Séminaire de Probabilités XXVII*, volume 1557 of *Lecture Notes in Mathematics*, pages 30–32. Springer Berlin Heidelberg, 1993.

[BCJR09] T. R. Bielecki, S. Crépey, M. Jeanblanc, and M. Rutkowski. Defaultable game options in a hazard process model. *Journal of Applied Mathematics and Stochastic Analysis*, pages Art. ID 695798, 33, 2009.

[BDH+03] P. Briand, B. Delyon, Y. Hu, E. Pardoux, and L. Stoica. L^p solutions of backward stochastic differential equations. *Stochastic Processes and their Applications*, 108(1):109–129, 2003.

[Bec06] D. Becherer. Bounded solutions to backward SDE's with jumps for utility optimization and indifference hedging. *The Annals of Applied Probability*, 16(4):2027–2054, 2006.

[BEK13] P. Barrieu and N. El Karoui. Monotone stability of quadratic semimartingales with applications to unbounded general quadratic BSDEs. *The Annals of Probability*, 41(3B):1831–1863, 2013.

[Ben84] A. Bensoussan. On the theory of option pricing. *Acta Applicandae Mathematica*, 2(2):139–158, 1984.

[BH08] P. Briand and Y. Hu. Quadratic BSDEs with convex generators and unbounded terminal conditions. *Probability Theory and Related Fields*, 141(3-4):543–567, 2008.

[BHM13] B. Björk, K. Holger, and C. Munk. Solving constrained consumption–investment problems by simulation of artificial market strategies. *Management Science*, 59(2):485–503, 2013.

[Bis73] J.-M. Bismut. Conjugate convex functions in optimal stochastic control. *Journal of Mathematical Analysis and Applications*, 44:384–404, 1973.

[BJR04] T. R. Bielecki, M. Jeanblanc, and M Rutkowski. Modeling and valuation of credit risk. In *Stochastic methods in finance*, volume 1856 of *Lecture Notes in Mathematics*, pages 27–126. Springer, Berlin, 2004.

[BK10] E. Bayraktar and R. Kravitz. On the stability of utility maximization problems. arXiv:1010.4322, 2010.

[BNS01] O. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein–Uhlenbeck-based models and some of their uses in financial economics. *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, 63(2):167–241, 2001.

[BPTZ15] B. Bouchard, D. Possamaï, X. Tan, and C. Zhou. A unified approach to a priori estimates for supersolutions of BSDEs in general filtrations. To appear in Annales de l'Institut Henri Poincaré B, 2015.

[BR02] T. R. Bielecki and M. Rutkowski. *Credit risk: modelling, valuation and hedging.* Springer Finance. Springer-Verlag, Berlin, 2002.

[BSJ04] C. Blanchet-Scalliet and M. Jeanblanc. Hazard rate for credit risk and hedging defaultable contingent claims. *Finance & Stochastics*, 8(1):145–159, 2004.

[BT04] B. Bouchard and N. Touzi. Discrete-time approximation and Monte-Carlo simulation of backward stochastic differential equations. *Stochastic Processes and their Applications*, 111(2):175–206, 2004.

[BY12] E. Bayraktar and S. Yao. Quadratic reflected BSDEs with unbounded obstacles. *Stochastic Processes and their Applications*, 122(4):1155–1203, 2012.

[BZ08] C. Bender and J. Zhang. Time discretization and Markovian iteration for coupled FBSDEs. *The Annals of Applied Probability*, 18(1):143–177, 2008.

[BZ14] E. Bayraktar and Z. Zhou. On controller-stopper problems with jumps and their applications to indifference pricing of american options. *SIAM Journal on Financial Mathematics*, 5(1):20–49, 2014.

[CFL14] A. Capponi and J.-E. Figueroa-López. Dynamic portfolio optimization with a defaultable security and regime-switching. *Mathematical Finance*, 24(2):207–249, 2014.

[CGN15] S. Choukroun, S. Goutte, and A. Ngoupeyou. Mean-variance hedging under multiple defaults risk. *Stochastic Analysis and Applications*, 33(5):757–791, 2015.

[Cha09] J.-F. Chassagneux. A discrete-time approximation for doubly reflected BSDES. *Advances in Applied Probability*, 41(1):101–130, 2009.

[CJN12] D. Coculescu, M. Jeanblanc, and A. Nikeghbali. Default times, no-arbitrage conditions and changes of probability measures. *Finance & Stochastics*, 16(3):513–535, 2012.

[CJZ13] G. Callegaro, M. Jeanblanc, and B. Zargari. Carthaginian enlargement of filtrations. *ESAIM: Probability and Statistics*, 17:550–566, 2013.

[CM14] B. Chikvinidze and M. Mania. New proofs of some results on bounded mean oscillation martingales using backward stochastic differential equations. *Journal of Theoretical Probability*, 27(4):1213–1228, 2014.

[CMS80] C. S. Chou, P.-A. Meyer, and C. Stricker. Sur les intégrales stochastiques de processus prévisibles non bornés. In *Séminaire de Probabilités XIV 1978/79*, volume 784 of *Lecture Notes in Mathematics*, pages 128–139. Springer,Berlin-New York, 1980.

[CR06] L. Carassus and M. Rásonyi. Convergence of utility indifference prices to the superreplication price. *Mathematical Methods of Operations Research*, 64(1):145–154, 2006.

[CR16] J.-F. Chassagneux and A. Richou. Numerical simulation of quadratic BSDEs. *The Annals of Applied Probability*, 26(1):262–304, 2016.

[Csi75] I. Csiszar. I-divergence geometry of probability distributions and minimization problems. *The Annals of Probability*, 3(1):146–158, 1975.

[DDM79] C. Doléans-Dade and P.-A. Meyer. Inégalités de normes avec poids. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Mathematics*, pages 313–331. Springer, Berlin, 1979.

[Del02] F. Delarue. On the existence and uniqueness of solutions to FBSDEs in a non-degenerate case. *Stochastic Processes and their Applications*, 99(2):209–286, 2002.

[DHR11] F. Delbaen, Y. Hu, and A. Richou. On the uniqueness of solutions to quadratic BSDEs with convex generators and unbounded terminal conditions. *Annales de l'Institut Henri Poincaré Probabilités et Statistiques*, 47(2):559–574, 2011.

[DL16a] R. Dumitrescu and C. Labart. Numerical approximation of doubly reflected BSDEs with jumps and RCLL obstacles. *Journal of Mathematical Analysis and Applications*, 442(1):206–243, 2016.

[DL16b] R. Dumitrescu and C. Labart. Reflected scheme for doubly reflected BSDEs with jumps and RCLL obstacles. *Journal of Computational and Applied Mathematics*, 296:827–839, 2016.

[DM82a] C. Dellacherie and P.-A. Meyer. *Probabilities and potential. B*, volume 72 of *North-Holland Mathematics Studies*. North-Holland Publishing Co., Amsterdam, 1982.

[DM82b] C. Dellacherie and P.-A. Meyer. *Probabilities and Potential B: Theory of Martingales*. North-Holland Mathematics Studies. North-Holland Publishing Co., Amsterdam, 1982.

[DS94] F. Delbaen and W. Schachermayer. A general version of the fundamental theorem of asset pricing. *Mathematische Annalen*, 300(3):463–520, 1994.

[DS95] F. Delbaen and W. Schachermayer. The no-arbitrage property under a change of numéraire. *Stochastics and Stochastic Reports*, 53(3-4):213–226, 1995.

[EI18] N. Esmaeeli and P. Imkeller. American options with asymmetric information and reflected BSDE. *Bernoulli*, 24(2):1394–1426, 2018.

[EJY00] R. J. Elliott, M. Jeanblanc, and M. Yor. On models of default risk. *Mathematical Finance*, 10(2):179–195, 2000.

[EK81] N. El Karoui. Les aspects probabilistes du cône trole stochastique. In *Ecole d'Eté de Probabilités de Saint-Flour IX-1979*, volume 876 of *Lecture Notes in Mathematics*, pages 73–238. Springer, Berlin New-York, 1981.

[EKH03] N. El Karoui and S. Hamadène. BSDEs and risk-sensitive control, zero-sum and nonzero-sum game problems of stochastic functional differential equations. *Stochastic Processes and their Applications*, 107(1):145–169, 2003.

[EKJJ10] N. El Karoui, M. Jeanblanc, and Y. Jiao. What happens after a default: The conditional density approach. *Stochastic Processes and their Applications*, 120(7):1011–1032, 2010.

[EKJJ15a] N. El Karoui, M. Jeanblanc, and Y. Jiao. Density approach in modeling successive defaults. *SIAM Journal on Financial Mathematics*, 6(1):1–21, 2015.

[EKJJ15b] N. El Karoui, M. Jeanblanc, and Y. Jiao. Dynamics of multivariate default system in random environment. Forthcoming in Stochastic Processes and Applications, 2015.

[EKKP+97] N. El Karoui, C. Kapoudjian, E. Pardoux, S. Peng, and M.-C. Quenez. Reflected solutions of backward SDE's, and related obstacle problems for PDEs. *The Annals of Applied Probability*, 25(2):702–737, 1997.

[EKMN16] N. El Karoui, A. Matoussi, and A. Ngoupeyou. Quadratic exponential semimartingales and application to BSDEs with jumps. arXiv:1603.06191, 2016.

[EKPQ97a] N El Karoui, E. Pardoux, and M.-C. Quenez. Reflected backward SDEs and American options. In *Numerical Methods in Finance*, volume 13 of *Publications of the Newton Institute*, pages 215–231. Cambridge University Press, 1997.

[EKPQ97b] N. El Karoui, S. Peng, and M. C. Quenez. Backward stochastic differential equations in finance. *Mathematical Finance*, 7(1):1–71, 1997.

[EKQ95] N. El Karoui and M-C. Quenez. Dynamic programming and pricing of contingent claims in an incomplete market. *SIAM Journal on Control and Optimization*, 33(1):29–66, 1995.

[EKR00] N. El Karoui and R. Rouge. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.

[Eme79] M. Emery. Une topologie sur l'espace des semimartingales. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 721 of *Lecture Notes in Mathematics*, pages 260–280. Springer, Berlin, 1979.

[Ess08] E.H. Essaky. Reflected backward stochastic differential equation with jumps and RCLL obstacle. *Bulletin des Sciences Mathématiques*, 132(8):690–710, 2008.

[FI93] H. Föllmer and P. Imkeller. Anticipation cancelled by a Girsanov transformation: a paradox on Wiener space. *Annales de l'Institut Henri Poincaré. Probabilités et Statistiques*, 29(4):569–586, 1993.

[FI13] A. Fromm and P. Imkeller. Existence, uniqueness and regularity of decoupling fields to multidimensional fully coupled FBSDEs. arXiv:1310.0499, 2013.

[FMW12] C. Frei, M. Mocha, and N. Westray. BSDEs in utility maximization with BMO market price of risk. *Stochastic Processes and their Applications*, 122(6):2486–2519, 2012.

[Fon15] C. Fontana. The strong predictable representation property in initially enlarged filtration. arXiv:1508.03282., 2015.

[Fre13] C. Frei. Convergence results for the indifference value based on the stability of BSDEs. *Stochastics. An International Journal of Probability and Stochastic Processes*, 85(3):464–488, 2013.

[Fri00] M. Frittelli. The minimal entropy martingale measure and the valuation problem in incomplete markets. *Mathematical Finance*, 10(1):39–52, 2000.

[GIO+15] M. Grigorova, P. Imkeller, E. Offen, Y. Ouknine, and M.-C. Quenez. Reflected BSDEs when the obstacle is not right-continuous and optimal stopping. To appear in Annals of Applied Probability, 2015.

[GK00] T. Goll and J. Kallsen. Optimal portfolios for logarithmic utility. *Stochastic Processes and their Applications*, 89(1):31–48, 2000.

[GP97] A. Grorud and M. Pontier. Comment détecter le délit d'initiés? *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 324(10):1137–1142, 1997.

[GP98] A. Grorud and M. Pontier. Insider trading in a continuous time market model. *International Journal of Theoretical and Applied Finance*, 01(03):331–347, 1998.

[GP99] A. Grorud and M. Pontier. Probabilités neutres au risque et asymétrie d'information. *Comptes Rendus de l'Académie des Sciences. Série I. Mathématique*, 329(11):1009–1014, 1999.

[GR02] P. Grandits and T. Rheinländer. On the minimal entropy martingale measure. *The Annals of Probability*, 30(3):1003–1038, 2002.

[GT11] P. Grandits and S. Thonhauser. Risk averse asymptotics in a Black-Scholes market on a finite time horizon. *Mathematical Methods of Operations Research*, 74(1):21–40, 2011.

[Ham02] S. Hamadène. Reflected BSDEs with discontinuous barrier and application. *Stochastics and Stochastic Reports*, 74(3–4):571–596, 2002.

[Hes93] S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6(2):327–343, 1993.

[HHI+14] U. Horst, Y. Hu, P. Imkeller, A. Réveillac, and J. Zhang. Forwad-backward systems for expected utility maximization. *Stochastic Processes and their Applications*, 124(5):1813–1848, 2014.

[HIM05] Y. Hu, P Imkeller, and M. Müller. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1972, 2005.

[HKS05] J. Hugonnier, D. Kramkov, and W. Schachermayer. On utility-based pricing of contingent claims in incomplete markets. *Mathematical Finance*, 15(2):203–212, 2005.

[HL00] S. Hamadène and J.-P. Lepeltier. Reflected BSDEs and mixed game problem. *Stochastic Processes and their Applications*, 85(2):177–188, 2000.

[HO16] S. Hamadène and Y. Ouknine. Reflected backward SDEs with general jumps. *Theory of Probability & Its Applications*, 60(2):263–280, 2016.

[HP12] S. Hamadene and A. Popier. $\mathbb{L}^p$-solutions for reflected backward stochastic differential equations. *Stochastics and Dynamics*, 12(2):1150016, 2012.

[HWY92] S. He, J. Wang, and J. Yan. *Semimartingale Theory and Stochastic Calculus.* Beijing; CRC Press, 1992.

[IDRZ10] P. Imkeller, G. Dos Reis, and J. Zhang. Results on numerics for FBSDE with drivers of quadratic growth. In *Contemporary Quantitative Finance*, pages 159–182. Springer, 2010.

[IJL16] B. Iftimie, M. Jeanblanc, and T. Lim. Optimization problem under change of regime of interest rate. *Stochastics and Dynamics*, 16(5):1650015–31, 2016.

[IN17a] P. Imkeller and V. Nzengang. FBSDEs related to utility maximization: analysis of solutions and risk aversion asymptotics. working paper, 2017.

[IN17b] P. Imkeller and V. Nzengang. Optimal stopping problem in a progressively enlarged filtration: A two step decomposition approach. Working paper, 2017.

[IN17c] P. Imkeller and V. Nzengang. Reflected BSDEs in a progresssively enlarged filtration: a two step approach. Working paper, 2017.

[IN17d] P. Imkeller and V. Nzengang. Utility maximization problem: Integrability properties and stability analysis of the solution. Working paper, 2017.

[ISS79] M. Izumisawa, T. Sekiguchi, and Y. Shiota. Remark on a characterization of BMO-martingales. *Tôhoku Mathematical Journal*, 31(3):281–284, 1979.

[Jac80] J. Jacod. Intégrales stochastiques par rapport à une semimartingale vectorielle et changements de filtration. In *Séminaire de Probabilités XIV 1978/79*, volume 784 of *Lecture Notes in Mathematics*, pages 161–172. Springer, Berlin, 1980.

[Jac85] J. Jacod. Grossissement initial, hypothèse (H') et théorème de Girsanov. In *Grossissements de filtrations: exemples et applications*, volume 1118 of *Lecture Notes in Mathematics*, pages 15–35. 1985.

[Jeu79] T. Jeulin. Grossissement d'une filtration et applications. In *Séminaire de Probabilités, XIII (Univ. Strasbourg, Strasbourg, 1977/78)*, volume 13 of *Lecture Notes in Mathematics*, pages 574–609. Springer, Berlin-New York, 1979.

[Jeu80] T. Jeulin. *Semi-Martingales et Grossissement d'une filtration*, volume 833 of *Lectures Notes in Mathematics.* Springer, Berlin, 1980.

[JKP13] Y. Jiao, I. Kharroubi, and H. Pham. Optimal investment under multiple defaults risk: a BSDE-decomposition approach. *The Annals of Applied Probability*, 23(2):455–491, 2013.

[JLC09a] M. Jeanblanc and Y. Le Cam. Immersion property and credit risk modelling. In *Optimality and Risk-Modern Trends in Mathematical Finance*, pages 99–131. Springer,Berlin, 2009.

[JLC09b] M. Jeanblanc and Y. Le Cam. Progressive enlargement of filtrations with initial times. *Stochastic Processes and their Applications*, 119(8):2523–2543, 2009.

[JMN12] M. Jeanblanc, A. Matoussi, and A. Ngoupeyou. Robust utility maximization problem in a discontinuous filtration. arXiv:1201.2690, 2012.

[JMPR15] M. Jeanblanc, T. Mastrolia, D. Possamaï, and A. Réveillac. Utility maximization with random horizon: A BSDE approach. *International Journal of Theoretical and Applied Finance*, 18(07), 2015.

[JN04] E. Jouini and C. Napp. Convergence of utility functions and convergence of optimal strategies. *Finance & Stochastics*, 8(1):133–144, 2004.

[JP11] Y. Jiao and H. Pham. Optimal investment with counterparty risk: a default-density model approach. *Finance & Stochastics*, 15(4):725–753, 2011.

[JS03] J. Jacod and A.N. Shirayaev. *Limit theorems for stochastic processes*, volume 288 of *Grundlehren der mathematischen Wissenschaften.* Springer-Verlag, Berlin, 2003.

[JY78] T. Jeulin and M. Yor. Grossissement d'une filtration et semi-martingales : formules explicites. In *Séminaire de Probabilités, XII (Univ. Strasbourg, Strasbourg, 1976/1977)*, volume 649 of *Lecture Notes in Mathematics*, pages 78–97. Springer,Berlin, 1978.

[Kal00] J. Kallsen. Optimal portfolios for exponential Lévy processes. *Mathematical Methods of Operations Research*, 51(3):357–374, 2000.

[Kar88] I. Karatzas. On the pricing of american options. *Applied Mathematics and Optimization*, 17(1):37–60, 1988.

[Kar10] C. Kardaras. The continuous behavior of the numéraire portfolio under small changes in information structure, probabilistic views and investment constraints. *Stochastic Processes and their Applications*, 120(3):331–347, 2010.

[Kaz94] N. Kazamaki. *Continuous Exponential Martingales and BMO*, volume 1579 of *Lecture Notes in Mathematics.* Springer-Verlag, Berlin, 1994.

[Kch11] Y. Kchia. *Semimartingales et Problématiques Récentes en Finance Quantitative.* PhD thesis, Ecole polytechnique, 2011.

[KK98] I. Karatzas and S. G. Kou. Hedging american contingent claims with constrained portfolios. *Finance & Stochastics*, 2(3):215–258, 1998.

[KK04] R. Korn and H. Kraft. On the stability of continuous-time portfolio problems with stochastic opportunity set. *Mathematical Finance*, 14(3):403–414, 2004.

[KL12] I. Kharroubi and T. Lim. Progressive enlargement of filtrations and backward stochastic differential equations with jumps. *Journal of Theoretical Probability*, pages 1–42, 2012.

[Kli12] T. Klimsiak. Reflected BSDEs with monotone generator. *Electronic Journal of Probability*, 17(107):1–25, 2012.

[Kli13] T. Klimsiak. BSDEs with monotone generator and two irregular reflecting barriers. *Bulletin des Sciences Mathématiques*, 137(3):268–321, 2013.

[Kli15] T. Klimsiak. Reflected BSDEs on filtered probability spaces. *Stochastic Processes and their Applications*, 125(11):4204–4241, 2015.

[KLP13] Y. Kchia, M. Larsson, and P. Protter. Linking progressive and initial filtration expansions. In *Malliavin Calculus and Stochastic Analysis*, volume 34 of *Springer Proc. Math. Stat.*, pages 469–487. Springer, New York, 2013.

[KLQT02] M. Kobylanski, J.-P. Lepeltier, M.-C. Quenez, and S. Torres. Reflected BSDE with superlinear quadratic coefficient. *Probability and Mathematical Statistics-Wroclaw University*, 22(1):51–83, 2002.

[KLS87] I. Karatzas, J. P. Lehoczky, and S. E. Shreve. Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. *SIAM Journal on Control and Optimization*, 25(6):1557–1586, 1987.

[KLSX91] I. Karatzas, J. P. Lehoczky, S. E. Shreve, and G. Xu. Martingale and duality methods for utility maximization in an incomplete market. *SIAM Journal on Control and Optimization*, 29(3):702–730, 1991.

[KMK10] J. Kallsen and J. Muhle-Karbe. Utility maximization in affine stochastic volatility models. *International Journal of Theoretical and Applied Finance*, 13(03):459–477, 2010.

[Kob00] M. Kobylanski. Backward stochastic differential equations and partial differential equations with quadratic growth. *The Annals of Probability*, 28(2):558–602, 2000.

[KP15] T. Kruse and A. Popier. BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics An International Journal of Probability and Stochastic Processes*, 18:1–49, 2015.

[KQ12] M. Kobylanski and M-C. Quenez. Optimal stopping time problem in a general framework. *Electronic Journal of Probabality*, 17(72):1–28, 2012.

[Kra96] D. O. Kramkov. Optional decomposition of supermartingales and hedging contingent claims in incomplete security markets. *Probability Theory and Related Fields*, 105(4):459–479, 1996.

[KS98] I. Karatzas and S. E. Shreve. *Methods of mathematical finance*. Applications of mathematics (New York). Springer-Verlag, New York, 1998.

[KS99] D. Kramkov and W. Schachermayer. The asymptotic elasticity of utility function and optimal investment in incomplete markets. *The Annals of Applied Probability*, 9(3):904–950, 1999.

[KS02] Y. Kabanov and C. Stricker. On the optimal portfolio for the exponential utility maximization: remarks to the six-author paper. *Mathematical Finance*, 12(2):125–134, 2002.

[KS03] D. Kramkov and W. Schachermayer. Necessary and sufficient conditions in the problem of optimal investment in incomplete markets. *The Annals of Applied Probability*, 13(4):1504–1516, 2003.

[KS07] D. Kramkov and M. Sîrbu. Asymptotic analysis of utility-based hedging strategies for small number of contingent claims. *Stochastic Processes and their Applications*, 117(11):1606–1620, 2007.

[Kus99] S. Kusuoka. A remark on default risk models. In *Advances in Mathematical Economics*, volume 1 of *Advances in Mathematical Economics*, pages 69–82. Springer, Tokyo, 1999.

[KW16] D. Kramkov and K. Weston. Muckenhoupt's (A_p) condition and the existence of the optimal martingale measure. *Stochastic Processes and their Applications*, 126(9):2615–2633, 2016.

[KŽ11] C. Kardaras and G. Žitković. Stability of the utility maximization problem with random endowment in incomplete markets. *Mathematical Finance*, 21(2):313–333, 2011.

[Lan98] D. Lando. On cox processes and credit risky securities. *Review of Derivatives Research*, 2(2):99–120, 1998.

[Lar09] K. Larsen. Continuity of utility-maximization with respect to preferences. *Mathematical Finance*, 19(2):237–250, 2009.

[Lio14] A. Lionnet. Some results on general quadratic reflected BSDEs driven by a continuous martingale. *Stochastic Processes and their Applications*, 124(3):1275–1302, 2014.

[LL17] P. Luo and T. Ludovic. Solvability of coupled FBSDEs with diagonally quadratic generators. *Stochastics and Dynamics*, 17(6):1750043, 14, 2017.

[LLP80] E. Lenglart, D. Lepingle, and M. Pratelli. Présentation unifiée de certaines inégalités de la théorie des martingales. In *Séminaire de Probabilités XIV 1978/79*, volume 784 of *Lectures Notes in Mathematics*, pages 26–48. Springer, Berlin, 1980.

[LM84] J.-P. Lepeltier and M. A. Maingueneau. Le jeu de Dynkin en théorie générale sans l'hypothèse de Mokobodski. *Stochastics and Stochastic Reports*, 13(1-2):25–44, 1984.

[LMX05] J.-P. Lepeltier, A. Matoussi, and M. Xu. Reflected backward stochastic differential equations under monotonicity and general increasing growth conditions. *Advances in applied probability*, 37(1):134–159, 2005.

[LSM98] J.-P. Lepeltier and J. San Martin. Existence for BSDE with superlinear-quadratic coefficient. *Stochastics and Stochastics Reports*, 63(3-4):227–240, 1998.

[Lv07] K. Larsen and G. Žitković. Stability of utility-maximization in incomplete markets. *Stochastic Processes and their Applications*, 117(11):1642–1662, 2007.

[LX05] J.-P. Lepeltier and M. Xu. Penalization method for reflected backward stochastic differential equations with one r.c.l.l. barrier. *Statistics & Probability Letters*, 75(1):58–66, 2005.

[LX07] J.-P. Lepeltier and M. Xu. Reflected BSDE with quadratic growth and unbounded terminal value. arXiv:0711.0619, 2007.

[Mai78] M. A. Maingueneau. *Séminaire de Probabilités XII: Université de Strasbourg 1976/77*, chapter Temps d'arret optimaux et theorie generale, pages 457–467. Springer Berlin Heidelberg, 1978.

[Mer69] R. C. Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The Review of Economics and Statistics*, 51(3):247–257, 1969.

[Mer71] R. C. Merton. Optimum consumption and portfolio rules in a continuous-time model. 3(4):373–413, 1971.

[Mor09a] M.-A. Morlais. Quadratic BSDEs driven by a continuous martingale and applications to the utility maximization problem. *Finance & Stochastics*, 13(1):121–150, 2009.

[Mor09b] M.-A. Morlais. Utility maximization in a jump market model. *Stochastics: An International Journal of Probability and Stochastic Processes*, 81(1):1–27, 2009.

[MPX08] J. Mémin, S. Peng, and M. Xu. Convergence of solutions of discrete reflected backward SDE's and simulations. *Acta Mathematicae Applicatae Sinica, English Series*, 24(1):1–18, 2008.

[MS05] M. Mania and M. Schweizer. Dynamic exponential utility indifference valuation. *The Annals of Applied Probability*, 15(3):2113–2143, 08 2005.

[MT03a] M. Mania and R. Tevzadze. Backward stochastic PDE and imperfect hedging. *International Journal of Theoretical and Applied Finance*, 06(07):663–692, 2003.

[MT03b] M. Mania and R. Tevzadze. A unified characterization of q-optimal and minimal entropy martingale measures by semimartingale backward equations. *Georgian Mathematical Journal*, 10(2):289–310, 2003.

[MT10] M. Mania and R. Tevzadze. Backward stochastic PDEs related to the utility maximization problem. *Georgian Mathematical Journal*, 17(4):629–794, 2010.

[MW12] M. Mocha and N. Westray. Quadratic semimartingale BSDEs under an exponential moments condition. In *Séminaire de Probabilités XLIV*, volume 2046 of *Lecture Notes in Mathematics*, pages 105–139. Springer, Heidelberg, 2012.

[MW13] M. Mocha and N. Westray. The stability of the constrained utility maximization problem: A BSDE approach. *SIAM Journal on Financial Mathematics*, 4(1):117–150, 2013.

[MWZZ15] J. Ma, Z. Wu, D. Zhang, and J. Zhang. On well-posedness of forward–backward SDEs—A unified approach. *The Annals of Applied Probability*, 25(4):2168–2214, 2015.

[MY99] J. Ma and J. Yong. *Forward-backward stochastic differential equations and their applications*. Number 1702. Springer Science & Business Media, 1999.

[Ngo10] A. Ngoupeyou. *Optimisation des portefeuilles d'actifs soumis au risque de défaut*. PhD thesis, Université d'Evry Val d'Essonne, 2010.

[Nut10] M. Nutz. The opportunity process for optimal consumption and investment with power utility. *Mathematics and Financial Economics*, 3(3-4):139–159, 2010.

[Nut12a] M. Nutz. The Bellman equation for power utility maximization with semimartingales. *The Annals of Applied Probability*, 22(1):363–406, 2012.

[Nut12b] M. Nutz. Power utility maximization in constrained exponential Lévy models. *Mathematical Finance*, 22(4):690–709, 2012.

[Nut12c] M. Nutz. Risk aversion asymptotics for power utility maximization. *Probability Theory and Related Fields*, 152(3-4):703–749, 2012.

[Pen90] S. Peng. A general stochastic maximum principle for optimal control problems. *SIAM Journal on Control and Optimization*, 28(4):966–979, 1990.

[Pen93] S. Peng. Backward stochastic differential equations and applications to optimal control. *Applied Mathematics and Optimization*, 27(2):125–144, 1993.

[Pha10] H. Pham. Stochastic control under progressive enlargement of filtrations and applications to multiple defaults risk management. *Stochastic Processes and their Applications*, 120(9):1795–1820, 2010.

[PK96] I. Pikovsky and I. Karatzas. Anticipative portfolio optimization. *Advances in Applied Probability*, 28(4):1095–1122, 1996.

[PKTZ15] D. Possamai, N. Kazi-Tani, and C. Zhou. Quadratic BSDEs with jumps: a fixed-point approach. *Electronic Journal of Probability*, 20(66):1–28, 2015.

[Pli86] S. R. Pliska. A stochastic calculus model of continuous trading: optimal portfolios. *Mathematics of Operations Research*, 11(2):370–382, 1986.

[PP90] É. Pardoux and S. G. Peng. Adapted solution of a backward stochastic differential equation. *Systems & Control Letters*, 14(1):55–61, 1990.

[Pro04] P. Protter. *Stochastic integration and differential equations. 2nd ed.* Number 21 in Applications of Mathematics. Springer Berlin, 2004.

[PS06] G. Peskir and A. Shiryaev. *Optimal stopping and free-boundary problems.* Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 2006.

[QS13] M.-C. Quenez and A. Sulem. BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Processes and their Applications*, 123(8):3328 – 3357, 2013.

[QS14] M.-C. Quenez and A. Sulem. Reflected BSDEs and robust optimal stopping for dynamic risk measures with jumps. *Stochastic Processes and their Applications*, 124(9):3031–3054, 2014.

[Roc70] R. T. Rockafellar. *Convex analysis.* Princeton Mathematical Series, No. 28. Princeton University Press, Princeton, N.J., 1970.

[Roy06] M. Royer. Backward stochastic differential equations with jumps and related nonlinear expectations. *Stochastic Processes and their Applications*, 116(10):1358–1376, 2006.

[RS12] A. Rozkosz and L. Słomiński. $\mathbb{L}^p$-solutions of reflected BSDEs under monotonicity condition. *Stochastic Processes and their Applications*, 122(12):3875–3900, 2012.

[Rud76] W. Rudin. *Principles of mathematical analysis.* McGraw-Hill Book Co., New York-Auckland-Düsseldorf, third edition, 1976.

[Sat99] K.-I. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics.* Cambridge University Press, Cambridge, 1999.

[Sch95] M. Schweizer. On the minimal martingale measure and the Föllmer-Schweizer decomposition. *Stochastic Analysis and Applications*, 13(5):573–599, 1995.

[Sch04] W. Schachermayer. *Portfolio optimization in incomplete financial markets.* Cattedra Galileiana. Scuola Normale Superiore, Classe di Scienze, Pisa, 2004.

[Sek06] J. Sekine. On exponential hedging and related backward stochastic differential equations. *Applied Mathematics and Optimization*, 54(2):131–158, 2006.

[Son14] S. Song. Optional splitting formula in a progressively enlarged filtration. *ESAIM: Probability and Statistics*, 18:829–853, 2014.

[ST14] M. Santacroce and B. Trivellato. Forward backward semimartingale systems for utility maximization. *SIAM Journal on Control and Optimization*, 52(6):3517–3537, 2014.

[Str77] C. Stricker. Quasimartingales, martingales locales, semimartingales et filtration naturelle. *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, 39(1):55–63, 1977.

[Szi05] A. Szimayer. Valuation of American options in the presence of event risk. *Finance & Stochastics*, 9(1):89–107, 2005.

[Tev08] R. Tevzadze. Solvability of backward stochastic differential equations with quadratic growth. *Stochastic Processes and their Applications*, 118(3):503–515, 2008.

[Wei84] Z. Weian. Une remarque sur une même I. S. calculée dans deux filtrations. In *Séminaire de Probabilités XVIII*, volume 1059 of *Lecture Notes in Mathematics*, pages 172–173. Springer, Berlin, 1984.

[Xu11] M. Xu. Numerical algorithms and simulations for reflected backward stochastic differential equations with two continuous barriers. *Journal of Computational and Applied Mathematics*, 236(6):1137–1154, 2011.

[Zha04] J. Zhang. A numerical scheme for BSDEs. *The Annals of Applied Probability*, 14(1):459–488, 2004.

Selbständigkeitserklärung

Ich erkläre, dass ich die vorliegende Arbeit selbständig un nur unter Verwendung der angegebenen Literatur und Hilfsmittel angefertig habe.

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